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Abstract

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ALMOST-PERIODIC FUNCTIONS IN THE SPECTRAL ANALYSIS OF OPERATORS

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The present note is a continuation of work ([1]) and is intended to demonstrate a general scheme for applying harmonic analysis to the spectral theory of operators, using as an example one special class of operators in a Banach space.

Consider a completely continuous linear operator A in a weakly complete ([2], pp. 121-124), in particular, in a reflexive Banach space. Suppose that*

$$E_A \equiv \sup_{-\infty<t<\infty} \|e^{iAt}\| < \infty. \tag{1}$$

Take an arbitrary vector x and an arbitrary linear functional f, and form the function

$$f_x(t) = f(e^{iAt}x) \qquad (-\infty < t < \infty).$$

It is easy to see that $f_x(t)$ is an almost-periodic function (a.p.f.) of Bohr.** Indeed, first, by virtue of (1), $f_x(t)$ is bounded; second, its derivative is an a.p.f., since for any sequence $\{h_k\}_{k=1}^{\infty}$ of real numbers the family of functions

$$\dot{f}_x(t+h_k) = if(e^{iAt}Ae^{iAh_k}x)$$

is compact in $C(-\infty,\infty)$ by virtue of (1) and the complete continuity of the operator A.

Associate with the function $f_x(t)$ its Fourier–Bohr series

$$f_x(t) \sim \sum_{\lambda} c_\lambda(x;f)e^{i\lambda t}. \tag{2}$$

The coefficients of the series (2) are computed by the formula

$$c_\lambda(x,f) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f_x(t)e^{-i\lambda t}\,dt,$$

from which, taking into account the weak completeness of the space, there follows the existence of the weak limit

$$P_\lambda = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{iAt} e^{-i\lambda t}\, dt. \tag{3}$$

The operator P_λ is linear and bounded, $\|P_\lambda\| \leqslant E_A$, and moreover:

- a) $AP_\lambda = P_\lambda A = \lambda P_\lambda$;
- b) if $Ax = \mu x$, then $P_\lambda x = \delta_{\lambda,\mu} x$;
- c) $P_\lambda P_\mu = \delta_{\lambda,\mu} P_\mu$.

Property a) is obtained from the formula

$$AP_\lambda = P_\lambda A = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{iAt} A e^{-i\lambda t}\, dt = \lim_{T\to\infty} \frac{1}{2Ti} \int_{-T}^{T} (e^{iAt})' e^{-i\lambda t}\, dt$$

by integration by parts. To prove property b), it suffices to note that, if $Ax = \mu x$, then $e^{iAt} x = e^{i\mu t} x$, and * then apply formula (3). Finally, c) follows immediately from a) and b).

We see that the operator P_λ projects onto the subspace of solutions of the equation $Ax = \lambda x$. Therefore, if $\{\lambda_k\}_{k=1}^{\infty}$ is the sequence of eigenvalues of the operator A, then $P_\lambda = 0$ for $\lambda \neq \lambda_k$. Now there arises the formal expansion

$$x \sim \sum_{k=1}^{\infty} P_{\lambda_k} x \tag{4}$$

of an arbitrary vector x in the eigenvectors of the operator A. In this case, obviously,

$$Ax \sim \sum_{k=1}^{\infty} \lambda_k P_{\lambda_k} x. \tag{5}$$

To obtain nonformal results it is enough to use the completeness theorem for the theory of a.p.f. and theorems on summability and convergence of Fourier–Bohr series of a.p.f. We give examples of results of this kind.

1. *The system of eigenvectors of the operator A is complete.*

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* For this it is sufficient that $\|R_\lambda\| \leq 1/|\operatorname{Im}\lambda|$ for $\operatorname{Im}\lambda \neq 0$ ((3), pp. 289-294).

** All the theorems on a.p.f. used in the note are set out, for example, in the monograph (4).

Indeed, let the functional f vanish on all eigenvectors of the operator A. Then from (2), by virtue of the completeness theorem of the theory of a.p.f., it follows that $f_x(t) = 0$. Setting $t = 0$, we obtain $f(x) = 0$, i.e. $f = 0$.

2. *The formal expansion (4) uniquely determines the vector x.*

Indeed, if all terms of the series (4) are equal to zero, then, just as above, we obtain that $f(x) = 0$ for every functional f, i.e. $x = 0$.

3. *Denote by*

$$(\theta_{nk}) \quad (n = 1, 2, \dots;\ k = 1, 2, \dots, k_n) \tag{6}$$

the matrix of coefficients of the Fejér–Bochner kernels corresponding to the spectrum of the operator A, in such a way that θ_{nk} corresponds to the exponent λ_k. Then the series (4) is strongly summable by method (6) to the vector x.

Indeed, by virtue of the summability of the series (2) for $f_x(t)$ by the Fejér–Bochner method, the sequence of operators

$$Q_n = \sum_{k=1}^{k_n} \theta_{nk} P_{\lambda_k}$$

converges weakly to the identity operator E. Moreover, for each eigenvector x,

$$Q_{nx} = \theta_{nk} x$$

for sufficiently large n. Since $\theta_{nk} \to 1$ as $n \to \infty$, it follows that on the linear span of the eigenvectors, which is dense in the whole space,

$$\lim_{n\to\infty} \|Q_n x - x\| = 0.$$

Consequently, $Q_n \to E$ strongly.

4. *If the eigenvalues of the operator A are arithmetically linearly independent, then the series (4) converges strongly to the vector x, i.e., the eigenspaces of the operator A form a basis under an arbitrary numbering of the eigenvalues.*

Indeed, the series (2) in this case converges absolutely to $f_x(t)$. The transition from weak convergence to strong convergence is carried out analogously to the preceding case.

5. *If*

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* From this it is also clear that the eigenvalues of the operator A are real.

$$\sum_{k=1}^{\infty} \lambda_k^2 < \infty,$$

then the series (4) converges strongly to x for every "source-representable" vector x, i.e., for every vector of the form $x = Ah$.

Indeed, if $x = Ah$, then

$$x \sim \sum_{k=1}^{\infty} \lambda_k P_{\lambda_k} h,$$

and it remains to note that for any functional f

$$\sum_{k=1}^{\infty} |f(P_{\lambda_k} h)^2| = \sum_{k=1}^{\infty} |c_{\lambda_k}(h; f)|^2 < \infty.$$

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References

1. Yu. I. Lyubich, V. I. Matsaev, *DAN*, **131**, No. 1 (1960).
2. S. S. Banach, *Course of Functional Analysis*, 1948.
3. E. Hille, *Functional Analysis and Semi-Groups*, IL, 1951.
4. B. M. Levitan, *Almost Periodic Functions*, Moscow, 1953.

Note: Figure translations are in progress. See original paper for figures.