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Abstract

Full Text

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ON POLYNOMIALS LEAST DEVIATING FROM ZERO WITH AN ARBITRARY NUMBER OF PRESCRIBED COEFFICIENTS

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1. Let r numbers $1, a_1, \dots, a_{r-1}$ be prescribed. We seek a real polynomial

$$P_n(z;\ 1, a_1, \dots, a_{r-1}) = z^n + a_1 z^{n-1} + \dots + a_{r-1} z^{n-r+1} + \cdots$$

with prescribed leading coefficients, least deviating from zero on the interval $[-1, +1]$, and the deviation $E = E(1;\ a_1, \dots, a_{r-1})$ of this polynomial from zero. For $r = 1$ the solution is expressed in terms of trigonometric functions; in this case $P_n(z;\ 1) = 2^{-(n-1)} \cos n \arccos z$ (^{1}). For $r = 2$, $P_n(z;\ 1, a_1)$ is expressed in terms of elliptic functions (^{2}), and for $r = 3$, $P_n(z;\ 1, a_1, a_2)$ is expressed in terms of special automorphic functions (^{3}). Our solution for arbitrary r is based on the results of the work (^{4}), whose notation we shall use without reference.

2. The maximal set of points of the segment $[-1, +1]$ at which a polynomial $Q_n(z)$ assumes, with alternating signs, its deviation from zero on $[-1, +1]$ is usually called the Chebyshev set. We shall denote this set by $\mathrm{Tsh}(Q_n)$, and the number of points of this set by $N[\mathrm{Tsh}(Q_n)]$.

As is known, the polynomial $P_n(z;\ 1, a_1, \dots, a_{r-1})$ of least deviation is completely characterized by the inequality $N[\mathrm{Tsh}(P_n)] \geq n - r + 2$ (^{1}). We shall consider polynomials $P_n(z;\ 1, a_1, \dots, a_{r-1})$ in general position in the space $(a_1, \dots, a_{r-1})$, for which $N[\mathrm{Tsh}(P_n)] = n - r + 2$. If -1 and $+1 \in \mathrm{Tsh}(P_n)$, then this inequality is equivalent to the assertion that the number of arcs δ^ν of the set $\mu(P_n; E)$, intersecting $(-1, +1)$ and degenerating into points, is equal to $n - r$. If at least one of the points ± 1 does not belong to $\mathrm{Tsh}(P_n)$, then the number of such arcs is $n - r + 1$, and if both points $\pm 1 \in \mathrm{Tsh}(Q_n)$, then the number of such degenerate arcs is equal to $n - r + 2$. All parameters τ_ν corresponding to degenerate arcs δ^ν are equal to zero. We may assume that, for the polynomials $P_n(z;\ 1, a_1, \dots, a_{r-1})$, these are the only degenerate arcs δ^ν.*

Theorem 1. *The half-strip $T(P_n;\ E)$ contains $r - 1$, $r - 2$, or $r - 3$ free continuous parameters, depending on whether both points ± 1, one of these points, or neither of them belongs to the set $\mathrm{Tsh}(P_n)$.*

3. Let $[\alpha, \beta]$ be the largest interval containing $[-1, +1]$ and belonging to $\mu(P_n; E)$. There are four possibilities:

1) $\alpha = -1,\ \beta = +1$;

2) $\alpha < -1,\ \beta > +1$;

3) $\alpha = -1,\ \beta > +1$;

4) $\alpha < -1,\ \beta = +1$.
We shall consider each of these cases.

5) $\alpha = -1,\ \beta = +1$. In this case, inside $(-1, +1)$ there lie $n - r$ points of the set $\mathrm{Tsh}(P_n)$, and the strip $T(P_n;\ E)$ cannot correspond to one more polynomial $\widetilde{P}_n(z)$ least deviating from zero on $[-1, +1]$. If such a polynomial $\widetilde{P}_n(z)$ existed, then, according to the theorem from $(^4)$,

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* Between two neighboring points of the set $\mathrm{Tsh}(P_n)$ there can be only an even number of arcs δ^ν.

$$\widetilde{P}_n(z) = a^{-n} P_n(az + b), \quad \widetilde{E} = a^{-n} E, \quad a > 0,$$

whence it follows that $P_n(z)$ assumes $n - r + 2$ times the values $\pm\widetilde{E}$ on $[b - a,\ b + a]$. For a polynomial $P_n(z)$ in general position this is possible only when $a = 1,\ b = 0$. In case 1) the points $(a_1, \dots, a_{r-1})$ are in one-to-one correspondence with the parameters of the half-strips $T(P_n; E)$, and

$$P_n(z) = E \cos\varphi(z), \tag{1}$$

where $\varphi(z)$, as always, is the function conformally mapping the fundamental domain $F(P_n; E)$ onto $T(P_n; E)$:

$$\varphi(z) = n \int_c^z \frac{-\prod_1^q (z - x_\nu) \prod_1^{p-1+2m} (z - \xi_i) \prod_1^{l-m} (z - z^j)(z - \overline{z}^j)\, dz}{\left[(1 - z^2) \prod_1^q (z - \sigma_\nu)(z - \overline{\sigma}_\nu) \prod_1^{p-1} (z - c_i)(z - d_i) \prod_1^l (z - a_j)(z - \overline{a}_j)(z - b_j)(z - \overline{b}_j)\right]^{1/2}} - n\pi, \tag{2}$$

where (c_i, d_i) are the intervals entering into μ_i, and a_j, b_j are the endpoints of the component γ^j. The path of integration lies in the domain F. This hyperelliptic integral depends on $3(r - 1)$ unknown real parameters. If one adjoins the $r - 1$ unknown continuous parameters of the domain T and the deviation E, then in all there will be $4r - 3$ unknown real quantities. It can be shown that, for a certain choice of the integer parameters of the half-strips T, all these quantities are uniquely determined from equations analogous to the relations for determining the constants in the Schwarz-Christoffel integral and from the equations for the coefficients $a_1, a_2, \dots, a_{r-1}$.

2) $\alpha < -1,\ \beta > +1$. In this case the polynomial $P_n(z; 1, a_1, \dots, a_{r-1})$ in general position has $n-r+4$ deviation points on $[\alpha, \beta]$. Define the quantities a and b from the relations

$$-a + b = \alpha, \qquad a + b = \beta, \tag{3}$$

then the polynomial

$$\widehat{P}_n(z) = a^{-n} P_n(az + b) \tag{4}$$

has on $[-1, +1]$ $n - r + 4$ deviation points

$$-1 < \hat{\xi}_1 < \dots < \hat{\xi}_{n-r+2} < +1$$

and is the polynomial least deviating from zero with $r - 2$ prescribed leading coefficients. Obviously,

$$P_n(z) = a^n \widehat{P}_n\left(\frac{z-b}{a}\right), \qquad -1 < \frac{-1-b}{a} < \hat{\xi}_1, \quad 1 > \frac{1-b}{a} > \hat{\xi}_{n-r+2}. \tag{5}$$

Conversely, if $P_n(z)$ is a polynomial of the form (5), then $P_n(z)$ deviates least from zero on $[-1, +1]$ and $\alpha < -1,\ \beta > +1$. Thus, in this case the polynomials $P_n(z; 1, a_1, \dots, a_{r-1})$ are in one-to-one correspondence with the collection of half-strips $T(\widehat{P}_n; \widehat{E})$, depending on $r - 3$ continuous parameters, certain integer parameters, and two continuous parameters a and b satisfying the inequalities (5). Obviously,

$$P_n(z) = a^n \widehat{E} \cos\varphi\left(\frac{z-b}{a}\right), \tag{6}$$

where $\varphi(z)$ is the function mapping the fundamental domain $F(\widehat{P}_n; \widehat{E})$ onto the half-strip $T(\widehat{P}_n; \widehat{E})$.

3) $\alpha = -1,\ \beta < +1$. In this case the polynomial $P_n(z; 1, a_1, \dots, a_{r-1})$ in general position has on $[-1, \beta]$ $n - r + 3$ deviation points. The relations (3) become

$$-a + b = -1, \qquad a + b = \beta \quad \text{or} \quad a = \frac{\beta+1}{2}, \quad b = \frac{\beta-1}{2},$$

and the polynomial

$$\widehat{P}_n(z) = \left(\frac{2}{1+\beta}\right)^n P_n\left(\frac{1+\beta}{2} z + \frac{\beta-1}{2}\right) \tag{7}$$

has on $[-1, +1]$ $n - r + 3$ points of deviation $-1 < \hat{\xi}_1 < \dots < \hat{\xi}_{n-r+1} < +1$. Obviously,

$$P_n(z) = \left(\frac{1+\beta}{2}\right)^n \hat{P}_n\left(\frac{2}{1+\beta} z - \frac{\beta-1}{\beta+1}\right), \qquad \hat{\xi}_{n-r+1} < \frac{3-\beta}{1+\beta} < +1. \tag{8}$$

Conversely, if $P_n(z)$ is a polynomial of the form (8), then $P_n(z)$ deviates least from zero on $[-1,+1]$ and $\alpha = -1$, $\beta > +1$. In this case the polynomials $P_n(z; 1, a_1, \dots, a_{r-1})$ are in one-to-one correspondence with the collection of half-strips $T(\hat{P}_n; \hat{E})$, depending on $r-2$ continuous parameters, certain integer parameters, and the continuous parameter β, satisfying the inequality in (8). Obviously,

$$P_n(z) = \left(\frac{1+\beta}{2}\right)^n \hat{E} \cos\varphi\left(\frac{2}{1+\beta}z - \frac{\beta-1}{\beta+1}\right), \tag{9}$$

where $\varphi(z)$ maps $F(\hat{P}_n; \hat{E})$ onto $T(\hat{P}_n; \hat{E})$.

4) $\alpha < -1$, $\beta = +1$. In this case

$$P_n(z) = \left(\frac{1-\alpha}{2}\right)^n \hat{P}_n\left(\frac{2}{1-\alpha}z - \frac{1+\alpha}{1-\alpha}\right), \qquad -1 < -\frac{3+\alpha}{1-\alpha} < \hat{\xi}_1, \tag{10}$$

where $\hat{P}(z)$ is a polynomial having $n-r+3$ points of deviation $-1 < \hat{\xi}_1 < \cdots < \hat{\xi}_{n-r+2} < +1$ on $[-1,+1]$. Consequently,

$$P_n(z) = \left(\frac{1-\alpha}{2}\right)^n \hat{E} \cos\varphi\left(\frac{2}{1-\alpha}z - \frac{1+\alpha}{1-\alpha}\right), \tag{11}$$

where $\varphi(z)$ maps $F(\hat{P}_n; \hat{E})$ onto $T(\hat{P}_n; \hat{E})$.

4. As $z \to \infty$,

$$\lim \frac{\omega(z)}{z^n} = \frac{2}{E}$$

(see (4)). Since $\ln\omega(z) = -i\varphi(z)$, for the deviation E in case 1) we obtain the following formula:

$$\ln E = \lim_{z\to\infty}\left[\, n\ln z + i\varphi(z)\,\right] + \ln 2. \tag{12}$$

In cases 2), 3), and 4), this formula determines $\hat{E}$, respectively,

$$E = \alpha^n \hat{E}, \qquad E = \left(\frac{1+\beta}{2}\right)^n \hat{E}, \qquad E = \left(\frac{1-\alpha}{2}\right)^n \hat{E}. \tag{13}$$

5. The half-strip $T(P_n; E)$, in addition to continuous parameters, depends on certain combinations of integer parameters. A count shows that the number $M(n; r)$ of such different combinations in case 1) is equal to

$$M(n;r) = \sum_{p,q_1,q_2,l,m} p \binom{n-r+q_1}{q_1} \binom{m+p}{m} \binom{p+q_2-2}{q_2} R_{l,m,p+2q_1+q_2}, \quad (14)$$

where $1 \leq p,\ 0 \leq q_1,\ 0 \leq q_2$ (if $p = 1$, then $q_2 = 0$), $0 \leq m \leq l$, and

$$p + 2q_1 + q_2 + 2l = r; \quad (15)$$

here $2q_1$ is the number of arcs δ^ν on $(-1, +1)$, and q_2 is the number of the remaining arcs δ^ν. The quantity R is defined by formulas (3) and (4) from (4).

Cases 3) and 4) are different; therefore the total number of different integer combinations on which the polynomial $P_n(z; 1, a_1, \dots, a_{r-1})$ depends is equal to

$$M(n;r) + 2M(n;r-1) + M(n;r-2). \quad (16)$$

For $q_1 = q_2 = l = 0$, the fundamental domain $F(P_n; Q)$ reduces to the upper half-plane, and $T(P_n; E)$ to a half-strip with base $[-n\pi, 0]$, divided by $r-1$ cuts into r segments, one of which has length

$(n-r+1)\pi$, and the remaining $r-1$ intervals each have length π. In this case the function $\varphi(z)$ is expressed by the Schwarz-Christoffel formula

$$\varphi(z) = -in \int_{-1}^{z} \frac{(z-\xi_1)\dots(z-\xi_{r-1})\,dz}{\prod_{j=1}^{r}[(z-c_j)(z-d_j)]^{1/2}} + (\nu - 1 - n)\pi, \quad (17)$$

if the interval $[c_\nu, d_\nu]$ coincides with $[-1, +1]$ and $c_1 < d_1 < c_2 < \dots < c_r < d_r,\ d_j < \xi_j < c_{j+1}$. The constants c, d, ξ are found from the relations

$$n \int_{c_i}^{d_i} |f(z)|\,dz = \pi, \quad i = 1, \dots, \nu{-}1, \nu{+}1, \dots, r{-}1; \quad \int_{d_j}^{\xi_j} |f(z)|\,dz = \int_{\xi_j}^{c_{j+1}} |f(z)|\,dz \quad (18)$$

and from the expansion in a neighborhood of the point $z = \infty$

$$-\left(i\varphi(z) + n \ln z + \ln \frac{2}{E}\right) = \ln\left(1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots + \frac{a_{r-1}}{z^{r-1}}\right) + O\left(\frac{1}{z^r}\right). \quad (19)$$

We omit the description of the determination of $\varphi(z)$ in the general case, when $2q_2 + q_1 + 2l \neq 0$, for lack of space.

6. Examples. Consider $r = 1, 2, 3$. Only the case $\alpha = -1,\ \beta = +1$ is of interest.

I. $r = 1$. In this case $p = 1,\ q_1 = 0,\ q_2 = 0,\ l = m = 0$. $M(n;1) = 1$. The half-strip T does not depend on parameters and has no slits; F is a half-plane; $\varphi(z)$ maps F onto T in such a way that $[-1,+1]$ passes into the base of T, $[-n\pi, 0]$. Obviously, $\varphi(z) = n \arccos z$.

II. $r = 2$. From $p + 2q_1 + q_2 + 2l = 2$ it follows that $p = 2,\ q_1 = q_2 = l = 0$ (the case $p = q_1 = 1$ is forbidden); $M(n;2) = 2$. F is a half-plane, and the half-strip T depends only on one parameter—the length of a slit with base either at the point $-n\pi + (n-1)\pi$, or at the point $-n\pi + \pi$. Let, for definiteness, the first case occur. Then $\mu(P;E)$ consists of the intervals $[-1,+1]$ and $[c,d]$, $1 < c$, and

$$\varphi(z) = n \int_{-1}^{z} \frac{(z-\xi)\,dz}{\sqrt{(1-z^2)(z-c)(z-d)}} - n\pi, \qquad 1 < \xi < c.$$

The constants c, d and ξ are determined from (18) and $n(c + d - 2\xi) = -2a_1$; $\ln E =$

$$= n \int_{d}^{\infty} \frac{x-\xi}{[(x^2-1)(x-c)(x-d)]^{1/2}}\,dx + \ln 2 + n \ln d.$$

III. $r = 3$. From $p + 2q_1 + q_2 + 2l = 3$ there follow the cases:

1) $p = 3,\ q_1 = q_2 = l = 0$;
2) $p = 2,\ q_1 = l = 0,\ q_2 = 1$;
3) $p = 1,\ q_1 = 1,\ q_2 = l = 0$;
4) $p = 1,\ q_1 = q_2 = 0,\ l = 1,\ m = 0$;
5) $p = 1,\ q_1 = q_2 = 0,\ l = m = 1$. $M(n;3) = n + 9$.

7. In the same way one solves the problem of the polynomial least deviating from zero on several prescribed intervals, with any number of leading fixed coefficients.

This method is also applicable to the solution of the problem of the polynomial least deviating from zero when not the leading r coefficients are prescribed, but any r linear relations between the coefficients of the polynomial.

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Note: Figure translations are in progress. See original paper for figures.