MATHEMATICS

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1960

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Abstract

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MATHEMATICS

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REGULAR T-SYSTEMS AND SOME QUESTIONS IN THE THEORY OF GENERALIZED POLYNOMIALS OF V. A. MARKOV

(Presented by Academician N. N. Bogolyubov on 6 VI 1960)

For greater definiteness and in order to shorten the formulations, here, whenever no explicit reservations are made, we shall have in mind **regular TM-systems*** $\{\varphi_\nu(x)\}_{\nu=0}^n$ $(a \le x \le b)$, which may also have fixed zeros (at one or both of the points a, b). However, with the exception of one point of more substantial difference in the supplementary part of the formulation of the generalization of the basic theorem of V. A. Markov, the other results extend without any essential changes to more general **regular T-systems** (including the periodic case).

1°. **Generalization of Vl. Markov' s problem and of his basic theorem.** The problem for the generalized polynomial

$$F(x) = \sum A_\nu \varphi_\nu(x)$$

$$\max_{a\le x\le b} |F(x)| = L[F] = L(A_0, \dots, A_n) = \min\ (= \rho) \tag{1}$$

under the condition

$$\omega[F] = A_0\alpha_0 + A_1\alpha_1 + \dots + A_n\alpha_n = 1 \qquad \left(\sum |\alpha_\nu| > 0\right) \tag{2}$$

always has a solution, unique or infinitely many-valued, with the usual general properties of the set of solutions for Chebyshev-type problems.

A direct generalization of the investigation carried out by E. Ya. Remez ([5]) in the case of Vl. Markov' s classical **rational-polynomial** problem ([6]) leads to the generalized basic theorem of Vl. Markov in the following form:

In order that a "nondegenerate" polynomial $\widetilde{F}(x) = \sum \widetilde{A}_\nu \varphi_\nu(x) \not\equiv \text{const}$, satisfying condition (2), with deviation points $x_0, \dots, x_{\chi-1}$ $(1 \le \chi \le n+1)$, be a solution of problem (1), it is necessary and sufficient that there hold identically with respect to $A_0, \dots, A_n$ a relation of the form

$$\omega[F] \equiv \sum_{s=0}^{\chi-1} r_s F(x_s) \tag{3}$$

under the essential additional condition

$$r_s\widetilde{F}(x_s) \geq 0 \qquad (s = 0, \dots, \chi - 1). \tag{4}$$

Those among the deviation points x_s of the solution polynomial $\widetilde{F}(x)$ for which $r_s \neq 0$ turn out to be **Chebyshev deviation points** ([2,5])

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* Systems of functions of Chebyshev (T-systems) or Chebyshev-Markov (TM-systems) ([1–3]) are called **regular** if the polynomials of the system $F(x) = \sum A_\nu \varphi_\nu(x) \not\equiv \text{const}$ have no more than $n+1$ points of (maximal) deviation (from zero) on $[a, b]$. General questions connected with the introduction of the concept of regularity of T-systems were the subject of our preliminary communication ([4]).

problem (1)–(2). Changing, if necessary, the numbering of the points x_s, one may assume that the Chebyshev points are the points of deviation $x_0, \dots, x_{q-1}$ $(1 \leq q \leq \varkappa)$.*

The identity relation (3) is equivalent to the system of equalities

$$\sum_{s=0}^{\varkappa-1} r_s\varphi_\nu(x_s) = \alpha_\nu \qquad (\nu = 0, \dots, n). \tag{3'}$$

This latter system, regarded as a system of $n+1$ equations for finding $r_0, \dots, r_{\varkappa-1}$, is certainly determined when $\varkappa = n+1$, and overdetermined when $\varkappa < n+1$. For a concrete specification of $\{\varphi_\nu\}_0^n$, $\{\alpha_\nu\}_0^n$, $\{x_s\}_0^{\varkappa-1}$, an appropriate application of the Gaussian procedure of successive eliminations always makes it possible here to establish the consistency of equations $(3')$, and in the first case simultaneously to find the very set $\{r_s\}_0^{\varkappa-1}$ (obviously uniquely determined) for the final verification of the sign relations (4).

For considerations of a general character, the above formulation of the generalized theorem of V. A. Markov needs supplementation: it is desirable also to have a corresponding more explicit form of expression of the condition for consistency of equations $(3')$.**

In the general case of regular T-systems, the required consistency condition would consist in the vanishing of all $\binom{n+1}{\varkappa+1}$ determinants of order $\varkappa + 1$ of the augmented coefficient matrix of the system of equations $(3')$. But in the case under consideration, when the subsystem $\{\varphi_\nu(x)\}_0^{\varkappa-1}$ is also certainly a T-system, the necessary and sufficient condition required for the consistency of $(3')$ reduces to only $n + 1 - \varkappa$ equalities, which, as is easy to see, may be represented in the form

$$\omega[\mathfrak{F}_\nu] = 0 \qquad (\nu = 0, \dots, n - \varkappa), \tag{6}$$

where

$$\mathfrak{F}_\nu(x) = D(\varphi_0(x_0), \varphi_1(x_1), \dots, \varphi_{\varkappa-1}(x_{\varkappa-1}), \varphi_{\varkappa+\nu}(x)) \qquad (\nu = 0, \dots, n-\varkappa). \quad (7)$$

Here, in the general case, we have an unavoidable feature of appreciable computational complication in comparison with the classical case $\{x^\nu\}_0^n$: in this latter case the equalities (6), whose left-hand sides are nonlinear functions of $x_0, \dots, x_{\varkappa-1}$, reduce to the equivalent system of equalities

$$\omega[x^\nu f] = 0 \qquad (\nu = 0, \dots, n-\varkappa) \qquad \left[f(x) = \prod_0^{\varkappa-1}(x - x_s) = \sum_0^{\varkappa-1} q_i x^i + x^\varkappa \right], \quad (8)$$

which are linear relations with respect to the coefficients q_i $(i = 0, \dots, \varkappa - 1)$ of the determining polynomial $f(x)$; in the general case we find no similar simplifications. Nevertheless, in the treatment of the corresponding fundamental questions—in particular

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* Let us note in passing that the same relations (3)–(4), with $|\widetilde{F}(x_s)| > 0$ $(s = 0, \dots, q-1)$ instead of $|\widetilde{F}(x_s)| = L$, characterize in a more general way $\{\widetilde{F}(x_s)\}_0^{q-1}$ as a limiting system of deviations (2), delivering the lower bound for ρ—in generalization of the well-known theorem of Vallée-Poussin.

** As for explicit expressions for the unknowns themselves $r_0, \dots, r_{\varkappa-1}$, determined from the first $\varkappa$ equations, in closer analogy with the formulations of V. A. Markov himself, in the case considered by us they may be given in the form

$$r_s = \omega[\lambda_s(x \mid \varphi_0, \dots, \varphi_{\varkappa-1};\ x_0, \dots, x_{\varkappa-1})] \qquad (s = 0, \dots, \varkappa - 1), \qquad (5)$$

where under the sign of the functional ω stands the corresponding fundamental polynomial of generalized Lagrange interpolation.

related to the further generalization of Vl. Markov' s problem to the case of several prescribed linear relations instead of one (2), the conditions (6) can be used in the same way as the conditions (8) in the case $\{x^\nu\}_0^n$.

2°. The condition ensuring the uniqueness of the solution of problem (1)–(2) in the case of T^*-systems. Many of the most commonly used T-systems are T^*-systems ([2]). For polynomials of such systems (in particular, therefore, for regular T^*- and TM^*-systems) a sufficient condition for uniqueness of the solution can here be given in the form ([5]) $q > \frac{1}{2}(n + \eta)$, where η is the number of Chebyshev points of deviation falling at the endpoints of the interval $[a, b]$.

3°. Connection with the correlative problem of generalized moments. In the case of power moments, the mentioned problem, in its connection with Vl. Markov' s problem, was first posed and investigated by Ya. A. Shokhatov ([7]; cf. also ([5]), p. 323). The correlative moment problem generalized as applied to (1)–(2) consists in finding a function of bounded variation $\sigma(x)$ satisfying the conditions

$$\int_a^b \varphi_\nu(x)\,d\sigma(x) = \alpha_\nu, \quad (\nu = 0, \dots, n); \qquad V_a^b(\sigma) = \min. \tag{9}$$

An argument analogous to that applied by E. Ya. Remez ([5]) in the case $\{x^\nu\}_0^n$ now leads, first of all, to the main result:

If problem (1)–(2) has at least one nondegenerate solution ($F^{(0)}(x) \not\equiv \text{const}$), then the unique solution of problem (9) is a step function $\sigma(x)$ with q jumps $r_0, \dots, r_{q-1}$ (cf. item 1°) at the Chebyshev points of deviation $x_0, \dots, x_{q-1}$ of problem (1)–(2), and

$$V_a^b(\sigma) = \int_a^b |d\sigma| = \sum_0^{q-1} |r_s| = \frac{1}{\rho}. \tag{10}$$

The situation will be somewhat different only in the case of a sequence $\alpha = (\alpha_0, \dots, \alpha_n)$ that is **strictly positive** with respect to the system of functions $\{\varphi_\nu(x)\}_0^n$, if this latter also possesses a **unit polynomial** ([4], §3) $F^*(x) = \sum A_\nu^* \varphi_\nu(x) \equiv 1$ (for greater uniformity of interpretation we may in general suppose $\sum A_\nu^* \alpha_\nu \geq 0$, replacing, if necessary, $\{\varphi_\nu(x)\}_0^n$ by $\{-\varphi_\nu(x)\}_0^n$). In this case the unique solution of problem (1)–(2) is degenerate: $F^{(0)}(x) = \text{const} = 1 : \sum A_\nu^* \alpha_\nu = F^*(x) : \sum A_\nu^* \alpha_\nu$, while problem (9) has an infinite set of solutions–nondecreasing $\sigma(x)$, among which there will certainly be $\sigma(x)$ with an infinite set of points of increase; the required minimal value $V_a^b(\sigma)$ in this case also turns out to be equal to $1/\rho$, as in ([10])*.

4°. Questions of constructing solutions. Direct and inverse problems of the theory of generalized polynomials of Vl. Markov. Long before V. A. Markov ([6]) created his algebraic treatment of problem (1)–(2) in the case $\{x^\nu\}_0^n$, P. L. Chebyshev (1854), by means of analysis, effectively constructed ([10a]) his famous **Chebyshev polynomials** $T_n(x)$–the most important special case of Vl. Markov' s rational polynomials, corresponding to $\alpha = (0, \dots, 0, 1)$ for $[a, b] = [-1, 1]$. Subsequently it was successively clarified ([10b,11]); ([6]), Chs. II and III; see also ([9b]), that the same polynomials $T_n(x)$ provide exact solutions of the innumerable–

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* With respect to closely related questions of considerable fundamental importance, see ([3,5,8,9a]).

of a numerical set of other problems of the type of Vl. Markov' s, including several genuinely classical ones. Along with this, in Vl. Markov himself ((⌒6), Ch. I) and in some later authors we also find, besides an analysis of the simplest typical cases that admit an exact solution (for example, $n = 2$; also $q = 1$ for any $n > 1$, and certain other cases of singular-positive sequences α), particular specially constructed examples in which one succeeds in obtaining an exact solution by first determining the set $\{x_s\}_0^{x-1}$ of points of deviation of the required rational polynomial—on the basis of trial use of relations (8) for relatively small conjectural values of x, or of relations derived from a correlative minimum problem of type (9) for a stepwise constant $\sigma(x)$ with $q \leq n+1$ jumps r_s (under various more or less conjectural assumptions concerning the distribution of the signs of these jumps and the value of the number q itself).

In fact, unlimited possibilities open up for constructing particular examples of exactly determined (rational and generalized) polynomials of Vl. Markov if one proceeds "from the other end," solving the inverse problem: it turns out that any arbitrarily prescribed polynomial

$$F(x) = \sum A_\nu \varphi_\nu(x) \quad (\nu = 0, \dots, n;\ a \leq x \leq b)$$

is the solution of problem (1)–(2) for a suitably chosen $\alpha = \bar{\alpha} = (\bar{\alpha}_0, \dots, \bar{\alpha}_n)$. Moreover, it is easy to arrange that the Chebyshev points of deviation constitute any nonempty subset specified in advance ($s = s_i,\ i = 1, \dots, q$) of the set $\{x_s\}_0^{x-1}$ of points of deviation of the given $F(x)$.

Indeed, after preliminarily defining α by the equalities $(3')$ for any set $\{r_s\}_0^{x-1}$ subject only to the conditions

$$\operatorname{sgn} r_s = \begin{cases} \operatorname{sgn} F(x_s), & s \in \{s_i\}, \\ 0, & s \notin \{s_i\}, \end{cases} \tag{11}$$

it remains only to put $\bar{\alpha} = \alpha : H$, where $H = \sum A_\nu \alpha_\nu = \sum r_{sF}(x_s) > 0$. It then follows that the inverse problem always has ∞^{x-1} solutions.

Such a mode of procedure can yield, above all, various interesting inequalities (cf. some examples in (⌒12)). If, on the contrary, one prescribes arbitrarily the data of the "direct" problem (1)–(2), then the possibility of effectively obtaining exact solutions (even in the case $\{x^\nu\}_0^n$) by the above-mentioned trial techniques is, undoubtedly, extremely limited. The genuinely general path for solving concrete problems (1)–(2), as well as analogous problems with one or several linear relations, in which $\varphi_0, \dots, \varphi_n$ may even be arbitrary linearly independent continuous functions on a compact set of any number of dimensions, consists in eliminating dependent parameters and applying general numerical methods of successive approximation ((⌒2), Chs. V–VII); we shall not dwell on this here in greater detail.

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Received
6 VI 1960

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Note: Figure translations are in progress. See original paper for figures.