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Abstract

Full Text

MATHEMATICAL PHYSICS

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ON A CONTACT THERMOCONVECTIVE PROBLEM

(Presented by Academician S. L. Sobolev, 17 VI 1960)

Below we consider the following thermoconvective problem, of the type of problems "with concentrated capacity" introduced by A. N. Tikhonov ⁽¹⁾:

$$\frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial z^2} = \frac{1}{a^2} \frac{\partial U}{\partial t}, \quad 0 < z < \infty, 0 < r < \infty, t > 0; \quad (1^1)$$

$$\frac{\partial^2 U}{\partial r^2} + \frac{1 - 2\nu}{r} \frac{\partial U}{\partial r} + \alpha \frac{\partial U}{\partial z} = \frac{\partial U}{\partial t}, \quad z = 0, 0 < r < \infty, t > 0; \quad (1^2)$$

$$U = 0, \quad 0 \leq z < \infty, 0 < r < \infty, t = 0; \quad (1^3)$$

$$U = T(t), \quad z = 0, r = 0, t > 0, \quad (1^4)$$

$$\lim_{z \rightarrow \infty} U = 0. \quad (1^5)$$

Such a formulation leads to the problem of determining the temperature field of a thermally anisotropic reservoir under heat injection through a unit well.* The presence in (1²) of the convective term

$$-\frac{2\nu}{r} \frac{\partial U}{\partial r}$$

makes it possible to introduce condition (1⁴) for $\nu > 0$. For $\nu \leq 0$ the formulation of such a condition is impossible ⁽³⁾.

We solve problem (1¹)–(1⁵) by the method of integral transforms. Put

$$V(r, z, p) = p \int_0^\infty e^{-pt} U(r, z, t) dt, \quad W(p) = p \int_0^\infty e^{-pt} T(t) dt. \quad (2)$$

Then we must have

$$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r} \frac{\partial V}{\partial r} + \frac{\partial^2 V}{\partial z^2} - \frac{p}{a^2} V = 0, \quad 0 < z < \infty, \quad 0 < r < \infty; \quad (3^1)$$

$$\frac{\partial^2 V}{\partial r^2} + \frac{1-2\nu}{r} \frac{\partial V}{\partial r} + \alpha \frac{\partial V}{\partial z} - pV = 0, \quad 0 < r < \infty, \quad z = 0; \quad (3^2)$$

$$V = W(p), \quad z = 0, \quad r = 0; \quad (3^3)$$

$$\lim_{z \rightarrow \infty} V = 0. \quad (3^4)$$

We seek V in the form

$$V(r, z, p) = \int_0^\infty f(s, p) \exp[-z\sqrt{s^2 + p/a^2}] J_0(rs) ds. \quad (4)$$

Here $J_0(z)$ is the Bessel function. The V thus formally defined satisfies conditions (3¹) and (3⁴). We require fulfillment of the remaining condi—

* The thermal conductivity of the reservoir in the direction perpendicular to the stratification is assumed infinitely large. A formulation close to the one given here for the one-dimensional process of heat injection was given by Loverier (2). first equation (3ⁱ). Substituting (4) into (3²), we formally obtain

$$\int_0^\infty f(s, p) \left\{ s^2 \left[\ddot{J}_0(rs) + \frac{1-2\nu}{rs} \dot{J}_0(rs) \right] - \left(p + \alpha \sqrt{s^2 + \frac{p}{a^2}} \right) J_0(rs) \right\} ds = 0. \quad (5)$$

But

$$\ddot{J}_0(rs) + \frac{1}{rs} \dot{J}_0(rs) = -J_0(rs), \quad \dot{J}_0(rs) = -J_1(rs).$$

Consequently, we must have

$$\int_0^\infty sr f^*(s, p) J_0(rs) ds - 2\nu \int_0^\infty sf(s, p) J_1(rs) ds = 0, \quad (6)$$

where

$$f^*(s, p) = \frac{f(s, p)}{s} \left(s^2 + p + \alpha \sqrt{s^2 + \frac{p}{a^2}} \right). \quad (7)$$

Multiplying (6) by $J_0(rx)$, integrating with respect to r from 0 to ∞ , and changing the order of integration, we find

$$\int_0^\infty r J_0(rx) dr \int_0^\infty f^*(s, p) J_0(rs) s ds - 2\nu \int_0^\infty s f(s, p) ds \int_0^\infty J_0(rx) J_1(rs) dr = 0. \quad (8)$$

The first integral in the left-hand side of (8) is a double Fourier-Bessel integral and therefore is formally equal to $f^*(x, p)$. The inner integral in the second term is the discontinuous Weber-Schafheitlin integral *:

$$\Gamma(\nu - \mu) \int_0^\infty t^{\mu - \nu + 1} J_\mu(at) J_\nu(bt) dt = \begin{cases} 2^{\mu - \nu + 1} a^\mu b^{-\nu} (b^2 - a^2)^{\nu - \mu + 1}, & b > a, \\ 0, & b < a, \end{cases}$$

in which one must set $\mu = 0$, $\nu = 1$, $b = s$, $a = x$. Consequently, it is equal to s^{-1} for $x < s$ and to zero for $x > s$. Thus, (8) reduces to the equation

$$f(x, p) \frac{x^2 + p + \alpha \sqrt{x^2 + p/a^2}}{x} - 2\nu \int_x^\infty f(s, p) ds = 0. \quad (9)$$

Put

$$\varphi(x, p) = \int_x^\infty f(s, p) ds. \quad (10)$$

From (3) and (4) it follows that

$$\varphi(0, p) = W(p). \quad (11)$$

By virtue of (9),

$$\frac{d\varphi}{dx} + \frac{2\nu x}{x^2 + p + \alpha \sqrt{x^2 + p/a^2}} \varphi = 0. \quad (12)$$

Integrating (12), taking into account the initial condition (11), and then differentiating, gives

$$f(x, p) = W(p) f_1(x, p), \quad (13)$$

* It is assumed that $\operatorname{Re} \nu > \operatorname{Re} \mu > -1$.

where

$$f_1(x, p) = \frac{2\nu x \exp[-2\nu\psi(x, p)]}{x^2 + p + \alpha\sqrt{x^2 + p/a^2}}; \quad (14^1)$$

$$\psi(x, p) = \frac{t_1}{t_1 - t_2} \ln \frac{t_1 - \sqrt{x^2 + p/a^2}}{t_1 - \sqrt{p/a^2}} - \frac{t_2}{t_1 - t_2} \ln \frac{t_2 - \sqrt{x^2 + p/a^2}}{t_2 - \sqrt{p/a^2}}; \quad (14^2)$$

$$t_1 = -\frac{\alpha}{2} + \sqrt{\frac{\alpha^2}{2} + p\frac{1-a^2}{a^2}}, \quad t_2 = -\frac{\alpha}{2} - \sqrt{\frac{\alpha^2}{4} + p\frac{1-a^2}{a^2}}. \quad (14^3)$$

We take $t_1 = 0$ for $p = 0$; $t_2 = -\alpha$ for $p = 0^*$. Note that in the special case $a^2 = 1$

$$f_1(x, p) = \frac{2\nu x(\alpha + \sqrt{p})^{2\nu}}{\sqrt{x^2 + p}(\alpha + \sqrt{x^2 + p})^{2\nu+1}}, \quad a^2 = 1. \quad (15)$$

Let $f_1(x, p) \exp[-z\sqrt{x^2 + p/a^2}]$ be the Laplace–Carson transform of $\Phi(x, z, t)$. Then, by virtue of (2), (4), and the convolution theorem, we find, under the assumption that the order of integration may legitimately be changed, that

$$U(r, z, t) = \int_0^\infty J_0(rx) dx \frac{\partial}{\partial t} \int_0^t T(t - \tau) \Phi(x, z, \tau) d\tau. \quad (16)$$

Thus, for the actual construction of the solution of the problem it is necessary only to carry out the construction of $\Phi(x, z, t)$.

First of all note that for the multivalued function $\sqrt{x^2 + p/a^2}$ one must take the branch which assumes the value $|x|$ at $p = 0$, since only in this case will V , defined according to (4), satisfy condition (3⁴) for $\operatorname{Re} p > 0$. We shall further regard the p -plane as cut along the negative real semiaxis, and the argument of p as varying from $-\pi$ on the lower edge of the cut to $+\pi$ on its upper edge. Under this convention $f_1(x, p)$ will have singularities at the points

$$p_1 = -a^2x^2, \quad p_2 = 0, \quad p_3 = a^2\alpha^2/4(a^2 - 1). \quad (17)$$

In addition to these points, $f_1(x, p)$ could have singularities at points that are roots of the equations $\sqrt{x^2 + p/a^2} - t_i = 0$, $i = 1, 2$, $x \geq 0$, i.e. at the roots of the equations $x^2 + p + \frac{\alpha}{a}\sqrt{a^2x^2 + p} = 0$ ($x > 0$; $x = 0$). But, by virtue of the

convention made on the choice of the branch of $\sqrt{a^2x^2 + p}$ and on the cut of the p -plane, these equations have no roots in the region under consideration. The points p_1 and p_2 are branch points, while p_3 is an essentially singular point**. Finally, note that by virtue of (14), $pf_1(x, p)$ remains uniformly bounded along any straight line $\sigma + i\tau$ ($-\infty < \tau < \infty$; $\sigma > p_3$). This, however, is sufficient for the representability of $f_1 \exp[-z\sqrt{x^2 + p/a^2}]$ in the form of a Laplace–Carson integral. Consequently,

$$\Phi(x, z, t) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} \exp \left[pt - z\sqrt{x^2 + \frac{p}{a^2}} \right] f_1(x, p) \frac{dp}{p}, \quad z \geq 0. \quad (18)$$

Let us consider the special case $a^2 = 1$, when $f_1(x, p)$ is defined according to (15) and, consequently, p_3 ceases to be a singular point. It is easy to see that

* For $\ln z$ the principal value is taken, $|\operatorname{Im} \ln z| \leq \pi$.

** The single-valuedness of $f_1(x, p)$ in a neighborhood of p_3 follows from the fact that, upon circling around p_3 , t_1 passes into t_2 and conversely (see (14²), (14³)).

that in this case the integral (18) can be transformed into the integral

$$\begin{aligned} \Phi(x, z, t) &= \frac{2\nu a^{2\nu} e^{-zx}}{(x+a)^{2\nu+1}} + \frac{2\nu x}{\pi} \int_0^x \frac{\exp[-\rho t - z\sqrt{x^2 - \rho}] R_0^{2\nu}}{\sqrt{x^2 - \rho} (a + \sqrt{x^2 - \rho})^{2\nu+1}} \frac{\sin 2\nu\varphi}{\rho} d\rho \\ &- \frac{2\nu x}{\pi} \int_{x^2}^{\infty} \frac{e^{-\rho t}}{\rho \sqrt{\rho - x^2}} \frac{R_0^{2\nu}}{R^{2\nu+1}} \cos[z\sqrt{\rho - x^2} - 2\nu\varphi + (2\nu + 1)\psi] d\rho. \end{aligned} \quad (19)$$

Here

$$\begin{aligned} R_0 &= \sqrt{a^2 + \rho}, & R &= \sqrt{a^2 - x^2 + \rho}, \\ \varphi &= \operatorname{arctg} \frac{\sqrt{\rho}}{a}, & \psi &= \operatorname{arctg} \frac{\sqrt{\rho - x^2}}{a}. \end{aligned} \quad (20)$$

$U(r, z, t)$, defined according to (16), (18) (or, respectively, (16), (20)), gives a formal solution of the problem. However, one can verify that all the operations leading to its construction are legitimate, which means that U indeed, and not only formally, gives the solution of the problem.

Let us note that

$$\lim_{t \rightarrow \infty} \Phi = \frac{2\nu a^{2\nu} e^{-zx}}{(x+a)^{2\nu+1}}. \quad (21)$$

Let

$$T(t) \equiv 1, \quad a^2 = 1. \quad (22)$$

In this case, from (16) and (21) it follows that

$$\bar{V}(r, z) = \lim_{t \rightarrow \infty} U(r, z, t) = \int_0^\infty \frac{2\nu a^{2\nu} e^{-zx}}{(x+a)^{2\nu+1}} J_0(rx) dx. \quad (23)$$

It is easy to verify that $\bar{V}(r, z)$ gives the solution of the stationary problem obtained from (1-5) in the case $\partial U/\partial t \equiv 0$, $\bar{T} = 1$. Thus, the solution of the nonstationary problem tends asymptotically to the solution of the corresponding stationary problem*.

Let us note that from (23), with the aid of integration by parts, it follows**

$$\bar{V}(r, 0) = 1 - \int_0^\infty \left(1 + \frac{\tau}{\rho}\right)^{-2\nu} J_1(\tau) d\tau, \quad \rho = ar. \quad (24)$$

The present work owes its origin to a conversation with A. N. Tikhonov. We take this opportunity to express our gratitude to him for discussing the question.

Bashkir State University
named after the 40th Anniversary of October

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CITED LITERATURE

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3. L. I. Rubinshtein, *DAN*, 135, No. 3 (1960).
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* This conclusion is nontrivial. Indeed, A. N. Tikhonov (1) indicated cases when problems with concentrated capacitance do not tend to a finite limit as $t \rightarrow \infty$.

** Recall that, in the interpretation indicated above, $\bar{V}(r, 0)$ denotes the temperature of the reservoir under the thermal influence through a single borehole

in the case of a stationary thermal state. The convective parameter ν is proportional to the rate of flow of the injection well, while the heat-transfer parameter a is equal to the ratio of the coefficient of thermal conductivity of the reservoir in the direction of its extent to the coefficient of thermal conductivity of the surrounding rocks.

Note: Figure translations are in progress. See original paper for figures.

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