EXPANSION IN EIGENFUNCTIONS OF CERTAIN NON-SELF-ADJOINT BOUNDARY-VALUE PROBLEMS

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Abstract

Full Text

MATHEMATICS

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EXPANSION IN EIGENFUNCTIONS OF CERTAIN NON-SELF-ADJOINT BOUNDARY-VALUE PROBLEMS

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1°. Let $a(x)$ and $b(x)$ be certain complex-valued functions of bounded variation on the interval $[0, l]$ $(0 < l < +\infty)$, continuous at the endpoints of this interval and such that the Riemann–Stieltjes integral exists

$$\int_0^l a(x)\, db(x). \tag{1}$$

Further, let the real functions $\Delta_i(x)$ $(0 \le x \le l,\ i = 1, 2, \dots, m)$ be differentiable and satisfy the conditions

$$\Delta_i(x) \ge 0, \qquad \Delta_i(l) < l, \qquad \Delta_i'(x) \le \theta < 1 \quad (0 \le x \le l;\ i = 1, 2, \dots, m), \tag{2}$$

and let the complex-valued functions $q_i(x) \in L_1(0, l)$ be identically zero, respectively, for $x - \Delta_i(x) < 0$ $(i = 1, 2, \dots, m)$. Finally, let the complex-valued kernel $K(x, t)$ $(0 \le t \le x \le l)$ satisfy the condition

$$\int_0^l \int_t^l |K(x, t)|\, dx\, dt < +\infty. \tag{3}$$

Denote

$$\varphi_i(x) \equiv x - \Delta_i(x) \quad (0 \le x \le l;\ i = 1, 2, \dots, m). \tag{4}$$

From conditions (2) it follows that there exist functions

$$\psi_i(x) = \varphi_i^{-1}(x) \equiv x + \Delta_i^*(x), \qquad \Delta_i^*(x) \ge 0 \tag{5}$$

$$(\varphi_i(0) \le x \le \varphi_i(l);\ i = 1, 2, \dots, m).$$

In the class of functions of bounded variation on $[0, l]$, consider the following nonhomogeneous eigenvalue problems:

Problem (A).

$$\frac{d}{dx}[y(x) - a(x)] + \sum_{i=1}^{m} q_i(x) y(x - \Delta_i(x)) + \int_0^x K(x,t) y(t)\, dt = \lambda y(x), \tag{6}$$

$$y(0) = \alpha, \tag{7}$$

$$\beta y(l) + \int_0^l y(x)\, db(x) = A_0. \tag{8}$$

Problem (A*).

$$-\frac{d}{dx}[z(x) + b(x)] + \sum_{i=1}^{m} q_i^*(x) z(x + \Delta_i^*(x)) + \int_x^l K(t,x) z(t)\, dt = \lambda z(x), \tag{6*}$$

$$\alpha z(0) + \int_0^l z(x)\, da(x) = A_0, \tag{7*}$$

$$z(l) = \beta, \tag{8*}$$

where α, β, and A_0 are certain complex constants and

$$q_i^*(x) = \begin{cases} 0, & \text{for } \varphi_i(l) < x < l, \\ q_i(\psi_i(x))\psi_i'(x), & \text{for } 0 \le x \le \varphi_i(l). \end{cases} \qquad (i = 1, 2, \dots, m) \tag{9}$$

Denote by $y(x, \lambda)$ the solution of problem (6), (7), and by $z(x, \lambda)$ the solution of problem (6*), (8*). These solutions exist, are unique* and are entire functions of λ.

It is established that:

1. The eigenvalues of problems (A) and (A^*) coincide and are the A_0-points of the entire function

$$\omega(\lambda) = \beta y(l, \lambda) + \int_0^l y(x, \lambda)\, db(x). \tag{10}$$

2. For arbitrary λ and μ the relation

$$\int_0^l y(x,\lambda)z(x,\mu)\,dx = \frac{\omega(\lambda)-\omega(\mu)}{\lambda-\mu} \tag{11}$$

holds.

To each zero λ_n of the function $\omega(\lambda) - A_0$ we assign two systems of functions:

$$\left.\frac{\partial^j y(x,\lambda)}{\partial\lambda^j}\right|_{\lambda=\lambda_n} \qquad (j = 0, 1, \dots, p_n - 1); \tag{12}$$

$$\left.\sum_{k=0}^{p_n-j-1} \frac{b^{(n)}_{p_n-j-k-1}}{k!\,j!}\frac{\partial^k z(x,\lambda)}{\partial\lambda^k}\right|_{\lambda=\lambda_n} \qquad (j = 0, 1, \dots, p_n - 1), \tag{12*}$$

where p_n is the multiplicity of the zero λ and

$$b_k^{(n)} = \left.\frac{1}{k!}\frac{d}{d\lambda^k}\left\{\frac{(\lambda-\lambda_n)^{p_n}}{\omega(\lambda)-A_0}\right\}\right|_{\lambda=\lambda_n} \qquad (k = 0, 1, \dots, p_n - 1). \tag{13}$$

Denote

$$|\lambda_0| = \min_{\omega(\lambda_n)=A_0} |\lambda| \qquad (\omega(\lambda_0) = A_0); \tag{14}$$

$$y_0(x) = y(x,\lambda_0), \qquad z_0(x) = \left.\sum_{k=0}^{p_0-1}\frac{b^{(0)}_{p_0-k-1}}{k!}\frac{\partial^k z(x,\lambda)}{\partial\lambda^k}\right|_{\lambda=\lambda_0}. \tag{15}$$

Number all the remaining functions of the form (12) and (12*) in the order of nondecreasing $|\lambda_n|$, so that each two j-th functions have identical numbers: for $\operatorname{Im}\lambda_n \geq 0$, positive numbers, and for $\operatorname{Im}\lambda_n < 0$, negative numbers. We denote the resulting system of functions by

$$\{y_k(x), z_k(x)\} \qquad (k = 0, \pm1, \pm2, \dots). \tag{16}$$

* For definiteness, all functions of bounded variation will be assumed continuous from the left.

According to Theorem 1 of paper ([1]), it follows from formula (11) that the system (16) is biorthogonal on $[0, l]$ in the sense

$$\int_0^l y_k(x)z_p(x)\,dx = \begin{cases} 0, & k \neq p, \\ 1, & k = p. \end{cases} \tag{17}$$

Thus, it is natural to call problems (A) and (A^*) **mutually adjoint**.

2°. In this subsection we give formulations of some results on expansion with respect to the system (16), when the functions $q_i(x)/\Delta_j'(x)$ $(i, j = 1, 2, \dots, m)$ have bounded variation on $[0, l]$, the kernel $K(x, t)$ has bounded variation on $[0, l]$ in one of its arguments, uniformly with respect to the other argument (for $t > x$ we put $K(x, t) \equiv 0$), and

$$\alpha\beta A_0 \neq 0. \tag{18}$$

In this case all eigenvalues of problems (A) and (A^*) lie in a certain strip

$$\sigma_1 \leq \operatorname{Re}\lambda \leq \sigma_2$$

$(\sigma_1 \leq \log|A_0|/|\alpha\beta| \leq \sigma_2)$. Let

$$a(x) = a_1(x) + a_2(x) + a_3(x),$$

$$b(x) = b_1(x) + b_2(x) + b_3(x), \tag{19}$$

where $a_1(x)$ and $b_1(x)$ are absolutely continuous functions; $a_2(x)$ and $b_2(x)$ are jump functions; $a_3(x)$ and $b_3(x)$ are singular functions.

Denote

$$A = \bigvee_0^l (a_2 + a_3), \qquad B = \bigvee_0^l (b_2 + b_3). \tag{20}$$

Theorem 1. *Suppose*

$$|\beta|A + |\alpha|B + AB \leq \min\{|A_0|, |\alpha\beta|\}. \tag{21}$$

Then, if $f(x) \in L_2(0, l)$, *then*

$$\lim_{n\to\infty} \int_0^l \left| f(x) - \sum_{k=-n}^{n} y_k(x) \int_0^l f(t) z_k(t)\, dt \right|^2 dx =$$

$$= \lim_{n\to\infty} \int_0^l \left| f(x) - \sum_{k=-n}^{n} z_k(x) \int_0^l f(t) y_k(t)\, dt \right|^2 dx = 0. \tag{22}$$

Before formulating propositions on pointwise convergence and on equiconvergence of expansions with respect to the system (16), we introduce some sets characterized by the function $a(x)$.

By $C\{a(x)\}$ we denote the set of all points of discontinuity of the function $a(x)$ on the interval $(0,l)$. A set $e \in C\{a(x)\}$ will be called a set of type $\widetilde{C}\{a(x)\}$ if for every $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|a(x) - a(t)| \leq \varepsilon$, if and only if $x \in e$, $t \in (0,l)$, and $|x-t| \leq \delta$. Obviously, every closed set $e \in C\{a(x)\}$ is a set of type $\widetilde{C}\{a(x)\}$.

Next, denote by $L\{a(x)\}$ the set of all those points $x \in C\{a(x)\}$ for which there exist constants $0 < \delta < \min(x, l-x)$ and $M(x) < +\infty$ such that $|a'(t)| \leq M(x)$ for $t \in (x-\delta(x), x+\delta(x))$.

Finally, a set $e \in L\{a(x)\}$ will be called a set of type $\widetilde{L}\{a(x)\}$ if, for $x \in e$, $\delta(x)$ and $M(x)$ may be taken independent of x.

Let us note that each function of bounded variation $a(x)$ obviously determines a nonempty set $C\{a(x)\}$, whereas a nonempty set $L\{a(x)\}$ may fail to exist.

Theorem 2. Let the functions $a(x)$ and $b(x)$ satisfy condition (21). Then:

1) If $f(x)$ is a function of bounded variation on $[0,l]$, then for $x \in C\{a(x)\}$

$$\frac{1}{2}[f(x+0) + f(x-0)] = \sum_{k=-\infty}^{\infty} y_k(x) \int_0^l f(t) z_k(t)\, dt, \tag{23}$$

where the series on the right converges uniformly on every set of the type $\widetilde{C}\{a(x)\}$.

2) If

$$f(x) \log \frac{x}{1+x} \in L_1(0,l), \tag{24}$$

then for $x \in L\{a(x)\}$

$$\lim_{n\to\infty} \left\{ \sum_{k=-n}^{n} y_k(x) \int_0^l f(t) z_k(t)\, dt - \frac{1}{\pi} \int_0^l f(t) \frac{\sin \frac{2\pi n}{l}(x-t)}{x-t}\, dt \right\} = 0, \tag{25}$$

where the limit is attained uniformly on every set of the type $\widetilde{L}\{a(x)\}$.

An analogous theorem is also valid for expansions with respect to the system $\{z_k(x)\}$ $(k = 0, \pm 1, \ldots)$.

Let us also note that, when condition (21) is fulfilled, all sufficiently large (in modulus) zeros of the function $\omega(\lambda) - A_0$ are simple. However, one can give examples where no zero of the function $\omega(\lambda) - A_0$ is simple, but Theorems 1 and 2 remain valid.

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Note: Figure translations are in progress. See original paper for figures.