ON THE LIMIT VALUES OF THE INTEGRALS

1960

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Abstract

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MATHEMATICS

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ON THE LIMIT VALUES OF THE INTEGRALS

$$\int_a^{\xi\pm 0} \Omega(t)\, d\sigma(t)$$

UNDER THE CONDITIONS OF A. A. MARKOV

(Presented by Academician S. N. Bernstein on 22 XII 1959)

In this note we mainly adhere to the terminology and notation adopted in ([1]).

1. If the point $S \equiv (s_0, s_1, \dots, s_n)$ belongs to the cone K (([1]), p. 26), then there exist representations ([1])

$$s_k = \int_a^b u_k(t)\, d\sigma(t) \qquad (k = 0, 1, \dots, n;\ d\sigma(t) \geq 0). \tag{1}$$

Under known restrictions (([1]); ([3]), p. 146), the greatest (respectively the least) value of the integral

$$I^+ = \int_a^{\xi+0} \Omega(t)\, d\sigma(t) \qquad \left(\text{respectively } \ I^- = \int_a^{\xi-0} \Omega(t)\, d\sigma(t)\right)$$

$(a < \xi < b)$ under conditions (1), where S is an interior point of K, is attained for $\sigma(t)$ giving the canonical representation of the sequence $\{s_k\}_0^n$ with a mass at the point ξ.

In the present note we study the behavior of canonical representations of index $n+2$ under variation of the moments $\{s_k\}_0^n$, and solve the question of the limit values of the integrals I^+ and I^- under condition (1), in which the point S is no longer fixed, but varies in a certain parallelepiped situated inside K. Such a formulation of the question may prove useful for the approximate calculation of the integrals I^+ and I^-, when only approximate (with excess and with deficiency) values of the moments (1) of the function $\sigma(t)$ are known.

We note that the restrictions imposed below on the functions $\{u_k(t)\}_0^n$ are certainly fulfilled for those systems of functions for which all minors of the matrix

$\|u_k(t_j)\|_{k,j=0}^n$ are positive for $a \le t_0 < t_1 < \cdots < t_n \le b$. This property is possessed, in particular, by the systems of functions $\{t^{\alpha_k}\}_0^n$, $\left\{\frac{1}{t+a_k}\right\}_0^n$

$(0 \le \alpha_0 < \alpha_1 < \cdots < \alpha_n;\ 0 < a \le t \le b)$, $\{e^{\alpha_k t}\}_0^n$ $(\alpha_0 < \alpha_1 < \cdots < \alpha_n;\ a \le t \le b)$.

The results obtained constitute a development of certain ideas of P. L. Chebyshev and A. A. Markov ((2), pp. 307 and 373; (3), pp. 76 and 146), to which, apparently, attention has not previously been paid.

2. In what follows the name "canonical representations" is applied only to representations of index $n+2$. The location of the growth points ξ_j of the canonical representation of the sequence $\{s_k\}_0^n$ (which determines an interior point K) with a mass at a prescribed point $\xi \in (a,b)$ depends on the po-

positions of ξ relative to ξ_j and $\bar{\xi}_j$—points of increase, respectively, of the lower and upper principal representations:

for $n = 2\nu - 1$

a) if $\bar{\xi}_\mu < \xi < \bar{\xi}_\mu$, then $\bar{\xi}_0 = a$ and $\bar{\xi}_j < \xi_j < \bar{\xi}_j$ $(j = 1, 2, \ldots, \nu)$;

b) if $\bar{\xi}_{\mu-1} < \xi < \xi_\mu$, then $\bar{\xi}_{j-1} < \bar{\xi}_j < \xi_j$ $(j = 1, 2, \ldots, \nu)$ and $\xi_{\nu+1} = b$;

for $n = 2\nu$

a) if $\xi_{\mu-1} < \xi < \bar{\xi}_\mu$, then $\xi_{j-1} < \xi_j < \bar{\xi}_j$ $(j = 1, 2, \ldots, \nu+1)$;

b) if $\bar{\xi}_\mu < \xi < \xi_\mu$, then $\xi_0 = a$, $\bar{\xi}_j < \xi_j < \xi_j$ $(j = 1, 2, \ldots, \nu)$ and $\xi_{\nu+1} = b$.

We shall agree to call a representation of type a) the **lower canonical** representation (LCR), and one of type b) the **upper canonical** representation (UCR).

3. Let the functions $\{u_k(t)\}_0^n$ be continuous on $[a,b]$, and let the determinant

$$\Delta \begin{pmatrix} u_0 & u_1 & \ldots & u_n \\ t_0 & t_1 & \ldots & t_n \end{pmatrix}$$

and all its minors of order n be positive for

$$a \le t_0 < t_1 < \cdots < t_n \le b.$$

The table below describes the behavior of canonical representations with mass at the fixed point $\xi = \xi_\mu$, when $(-1)^{k+1} s_k$ increases.

n	Representation	ξ_j, $j < \mu$	ξ_j, $j > \mu$	Mass at the point a	Mass at the point b	Mass at the point $\xi = \xi_\mu$
$n = 2\nu - 1$	LCR	decrease	increase	decreases	–	increases
$n = 2\nu - 1$	UCR	increase	decrease	–	increases	decreases
$n = 2\nu$	LCR	increase	decrease	–	–	decreases
$n = 2\nu$	UCR	decrease	increase	decreases	decreases	increases

Definition. The set of points $S \in K$ for which the given point $\xi \in (a, b)$ is a point of increase of the lower (respectively upper) principal representation will be denoted by $S(\xi)$ (respectively $S(\bar{\xi})$). We shall call $S(\xi)$ and $S(\bar{\xi})$ the ξ-principal surfaces.

Taking into account the behavior of the principal representations under an increase of $(-1)^{k+1}s_k$ ([4]), one easily obtains the following properties of the ξ-principal surfaces.

1) $S(\xi)$ and $S(\bar{\xi})$ consist of interior points of K and do not intersect.
2) If, when the point S moves along some curve, $(-1)^{k+1}s_k$ $(k = 0, 1, \dots, n)$ do not decrease, then this curve can intersect only one of the ξ-principal surfaces, and then only at one point.
3) Every parallelepiped with edges parallel to the coordinate axes and lying inside K can have common points with only one ξ-principal surface.

If $A \in K$, $B \in K$ and $(-1)^{k+1}a_k \le (-1)^{k+1}b_k$ $(k = 0, 1, \dots, n)$, then the whole parallelepiped

$$(-1)^{k+1}a_k \le (-1)^{k+1}x_k \le (-1)^{k+1}b_k \quad (k = 0, 1, \dots, n) \tag{2}$$

belongs to K ([4]).

4) The parallelepiped (2) does not intersect the ξ-principal surfaces if and only if the point ξ determines:

for $n = 2\nu - 1$ an VKP at the point A or an NKP at the point B;

for $n = 2\nu$ an NKP at the point A or a VKP at the point B.

Let us consider in detail the case $n = 2\nu - 1$. We begin with an interior point $S \in K$, for which $\xi = \xi_\mu$ determines an NKP. As $(-1)^{k+1}s_k$ increases, the points ξ_j move away from ξ, remaining in the intervals $(\xi_j, \bar{\xi}_j)$, with ξ_j and $\bar{\xi}_j$ moving toward one another ([4]); the mass at the point ξ increases; the mass at the point a decreases. This will continue until S falls on $S(\xi)$ or $S(\bar{\xi})$, or on the boundary of K.

a) If S falls on $S(\xi)$, then ξ_μ coincides with ξ; the points ξ_j lying to the left of ξ "collide" with ξ_j; the points ξ_j lying to the right of ξ are "caught up with" by the points $\bar{\xi}_j$; the mass at the point a disappears. With a further increase of $(-1)^{k+1}s_k$, the representation becomes a VKP. The points ξ_j begin to move toward the point ξ; the mass at the point ξ decreases; at the point b a mass appears and grows. If, in this process, S approaches a boundary point of K whose representation index is equal to n, then the points ξ_j lying to the left of ξ "catch up with" ξ_j; the points ξ_j lying to the right of ξ "catch up with" ξ_{j-1}; the mass at the point ξ disappears.

b) An analogous picture is obtained if, from its initial position, S falls on $S(\bar{\xi})$.

c) If, from its initial position, S approaches a boundary point K without falling on the ξ-principal surfaces, then ξ_j, ξ_j, and $\bar{\xi}_j$ merge simultaneously; the mass at the point a disappears; in this case the representation of the boundary point has a mass at ξ. Such boundary points are limiting for both ξ-principal surfaces.

4. Suppose now that the functions $\{u_k(t)\}_0^n$ and $\Omega(t)$, continuous in $[a, b]$, satisfy the following condition: the determinant

$$\Delta \begin{pmatrix} u_0 & u_1 & \dots & u_m & \Omega \\ t_0 & t_1 & \dots & t_m & t_{m+1} \end{pmatrix}$$

and all its minors of orders $(m+1)$, m, and $(m-1)$ are positive for

$$a \le t_0 < t_1 < \dots < t_{m+1} \le b \quad (m = 0, 1, \dots, n)$$

(this condition can be weakened).

Theorem 1. Let the parallelepiped (2), where $A \in K$ and $B \in K$, not intersect the ξ-principal surfaces. The greatest (least) value of the integral I^+ (respectively I^-) under the conditions

$$(-1)^{k+1}a_k \le (-1)^{k+1}\int_a^b u_k(t)\,d\sigma(t) \le (-1)^{k+1}b_k \quad (k = 0, 1, \dots, n;\ d\sigma(t) \ge 0) \tag{3}$$

is attained

$$\text{for } n = 2\nu - 1 \text{ at the point } A, \text{ if there } \xi_{\mu-1} < \xi < \xi_\mu, \text{ or}$$

$$\text{at the point } B, \text{ if there } \xi_\mu < \xi < \bar{\xi}_\mu;$$

$$\text{for } n = 2\nu \quad \text{at the point } A, \text{ if there } \xi_{\mu-1} < \xi < \bar{\xi}_\mu, \text{ or}$$

$$\text{at the point } B, \text{ if there } \bar{\xi}_\mu < \xi < \xi_\mu.$$

Theorem 2. Let the parallelepiped (2), where $A \in K$ and $B \in K$, intersect $S(\xi)$ or $S(\bar{\xi})$. The greatest (least) value of the integral

I^+ (respectively I^-) under conditions (3) is attained

for $n = 2\nu - 1$ at the point of intersection of $S(\bar{\xi})$ with the broken line $A_0 A_1 \dots A_{n+1}$ or
at the point of intersection of $S(\bar{\xi})$ with the broken line $B_0 B_1 \dots B_{n+1}$;

for $n = 2\nu$ at the point of intersection of $S(\bar{\xi})$ with the broken line $B_0 B_1 \dots B_{n+1}$ or
at the point of intersection of $S(\bar{\xi})$ with the broken line $A_0 A_1 \dots A_{n+1}$.

Here

$$A_i \equiv (a_0,\ a_1,\ \dots,\ a_{i-1},\ b_i,\ \dots,\ b_{n-1},\ b_n), \qquad A_0 \equiv B, \qquad A_{n+1} \equiv A,$$

$$B_i \equiv (b_0,\ b_1,\ \dots,\ b_{i-1},\ a_i,\ \dots,\ a_{n-1},\ a_n), \qquad B_0 \equiv A, \qquad B_{n+1} \equiv B.$$

In conclusion, let us note that in the case of the classical moment problem ($u_k(t) = t^k$), effective methods are known (see, for example, (1)) for finding the principal and canonical representations.

The author expresses his deep gratitude to M. G. Krein for his interest in this work.

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Received
22 XII 1959

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