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Abstract

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ESTIMATE OF THE GROWTH OF A SOLUTION OF A SYSTEM WITH NONHOMOGENEOUS BOUNDARY CONDITIONS AND THE PHRAGMÉN–LINDELÖF THEOREMS

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We shall consider the system

$$\frac{\partial u}{\partial y} = A\frac{\partial u}{\partial x} + Pu, \qquad B_1 u(x,0) + B_2 u(x,1) = f(x), \tag{1}$$

where A, P, B_1, B_2 are matrices of order n; u and f are vectors.

We impose the following restrictions on the matrices A, B_1, B_2:

1) Among the eigenvalues of the matrix A there are no purely real ones.

2) Denote by E^+, respectively E^-, the projection operator onto the sum of the invariant subspaces of the matrix A corresponding to the eigenvalues a_k for which $\operatorname{Im} a_k > 0$, respectively $\operatorname{Im} a_k < 0$, and require that

$$\det(B_1E^+ + B_2E^-) \neq 0, \qquad \det(B_1E^- + B_2E^+) \neq 0. \tag{2}$$

Conditions 1) and 2), generally speaking, do not ensure completeness of the eigenfunctions of the system

$$\frac{du}{dy} = (zA + P)u, \qquad B_1 u(0) + B_2 u(1) = 0. \tag{3}$$

With respect to the vector $f(x)$, assume that

$$3) \qquad \|f(x) - f(s)\| < |x - s|^{\alpha}(\varphi(x) + \varphi(s)), \qquad \alpha > 0, \tag{4}$$

where

$$\varphi(x) > 0, \qquad \lim_{x\to+\infty}\frac{\varphi'(x)}{\varphi(x)} = 0.$$

Theorem 1. If $\det(B_1 + B_2 e^{P+zA})$ has no purely imaginary zeros, then there exists a solution $u_0(x, y)$ of system (1) for which

$$u_0(x, y) = O(\varphi(x) + \|f(x)\|).$$

(It should be noted that the zeros of $\det(B_1+B_2 e^{P+zA})$ are eigenvalues of system (3).)

Theorem 2. If the greatest multiplicity of the purely imaginary zeros of $\det(B_1 + B_2 e^{P+zA})$ is equal to p, then there exists a solution of system (1) for which

$$u_0(x, y) = O\left(\varphi(x) + \|f(x)\| + x^{p-1} \int_0^x \|f(t)\| \, dt\right).$$

Theorem 3. If $\det(B_1+B_2 e^{P+zA})$ has one simple purely imaginary zero $z = i\lambda_0$, then there exists a solution of system (1) of the form

$$u_0(x, y) = e^{Py+i\lambda_0(x+Ay)} G_0 \int_0^x e^{-i\lambda_0 t} f(t) \, dt + O(\varphi(x) + \|f(x)\|),$$

where G_0 is a certain constant matrix of rank one.

The example given at the end of the article shows that in Theorems 1-3 one cannot dispense with restriction 3), even in the classical case of the system

Cauchy-Riemann (boundedness of a harmonic function does not imply boundedness of its conjugate).

Combining Theorems 1-3 with the results of the note [1], we obtain Phragmén-Lindelöf theorems for system (1). For example:

Theorem 4. *If* $\det(B_1 + B_2 e^{P+zA})$ *has no zeros in the strip* $0 \le \operatorname{Re} z < \beta$, $u(x, y)$ *is a solution of system (1) satisfying the condition* $u(x, y) = o(e^{\beta x})$, $x \to +\infty$, *then*

$$u(x, y) = O(\varphi(x) + \|f(x)\|).$$

We shall divide the proof of Theorems 1-3 into four lemmas.

Lemma 1. *Put*

$$M(z, y) = e^{(P+zA)y} (B_1 + B_2 e^{P+zA})^{-1}.$$

For

$$|\arg z \pm \pi/2| \le \eta, \qquad \eta = \frac{1}{2} \min_k \{|\arg(\pm a_k)|\},$$

$$\|M(z, y)\| < C e^{-\tau y(1-y)|z|}, \qquad 0 \le y \le 1.$$

Proof. Put $A(z) = A + \frac{1}{z}P$. It is clear that, for sufficiently large z, among the eigenvalues of the matrix $A(z)$ there are no purely real ones. If $E^+(z)$ and $E^-(z)$ denote for $A(z)$ the same as E^+ and E^- for A, then, obviously, $E^\pm(z) \to E^\pm$ as $z \to \infty$, and therefore

$$\det(B_1 E^\pm(z) + B_2 E^\mp(z)) \to \det(B_1 E^\pm + B_2 E^\mp).$$

Let, for definiteness, $z \to \infty$, $|\arg z - \pi/2| < \eta$. Then

$$E^-(z)e^{zA(z)} = O(e^{-\tau|z|}), \qquad E^+(z)e^{-zA(z)} = O(e^{-\tau|z|}), \qquad \tau > 0. \tag{5}$$

Since $M(z,y) = (B_1 e^{-yzA(z)} + B_2 e^{(1-y)zA(z)})^{-1}$, putting

$$D = B_1 E^-(z)e^{-yzA(z)} + B_2 E^+(z)e^{(1-y)zA(z)},$$

$$\varepsilon = B_1 E^+(z)e^{-yzA(z)} + B_2 E^-(z)e^{(1-y)zA(z)},$$

we can, by virtue of the identity $E^+(z) + E^-(z) = E$, write

$$M(z,y) = (D+\varepsilon)^{-1} = D^{-1}(E+\varepsilon D^{-1})^{-1} = D^{-1} - D^{-1}\varepsilon D^{-1}(E+\varepsilon D^{-1})^{-1}. \tag{6}$$

Using the identities $E^\pm(z)E^\mp(z) = 0$, $(E^\pm(z))^2 = E^\pm(z)$, and the commutativity of $E^\pm(z)$ and $A(z)$, we easily find

$$D = (B_1 E^-(z) + B_2 E^+(z))(E^-(z)e^{-yzA(z)} + E^+(z)e^{(1-y)zA(z)}),$$

$$D^{-1} = (E^-(z)e^{yzA(z)} + E^+(z)e^{-(1-y)zA(z)})(B_1 E^-(z) + B_2 E^+(z))^{-1},$$

$$\varepsilon D^{-1} = (B_1 E^+(z)e^{-zA(z)} + B_2 E^-(z)e^{zA(z)})(B_1 E^-(z) + B_2 E^+(z))^{-1}.$$

By (5) this gives us $D^{-1} = O(e^{-\tau y(1-y)|z|})$, $\varepsilon D^{-1} = O(e^{-\tau|z|})$, whence, with the aid of (6), we obtain the assertion of the lemma.

Lemma 2. *The residue of $M(z,y)e^{zx}$ in the pole $z = z_n$ of multiplicity p has the form $e^{z_n x}H_n(x,y)$, where $H_n(x,y)$ is a polynomial in x of degree $p-1$ and*

$$B_1 H_n(x,0) + B_2 H_n(x,1) = 0.$$

Proof. We have

$$\operatorname*{res}_{z=z_n} M(z,y)e^{zx} = \frac{1}{(p-1)!}\frac{d^{p-1}}{dz^{p-1}}(z-z_n)^p e^{zx} M(z,y)\bigg|_{z=z_n} = e^{z_n x}\sum_{k=0}^{p-1} x^{kH}_{n,k}(y),$$

and, since by definition $B_1 M(z,0) + B_2 M(z,1) = E$, it follows that

$$B_1 H_n(x,0) + B_2 H(x,1) = \frac{e^{-z_n x}}{(p-1)!}\frac{d^{p-1}}{dz^{p-1}}(z-z_n)^p e^{zx} E\bigg|_{z=z_n} = 0,$$

which proves the lemma.

Denote by $L^{\pm}_{\delta}$ the boundary of the angle $|\arg(\pm z-\delta)| < \pi/2-\eta_1$, where $\delta > 0$ and $0 < \eta_1 < \eta$ (see Lemma 1) are chosen so that between L^{+}_{δ} and L^{-}_{δ} lie ···

have only purely imaginary poles of $M(z, y)$. By L_0 we denote an arbitrary closed contour lying between L^{+}_{δ} and L^{-}_{δ} and containing inside itself the point $z = 0$ and all purely imaginary poles of $M(z, y)$.

Lemma 3. Denote

$$K(x,y) = \frac{1}{2\pi i}\int_{L^{-}_{\delta}} M(z,y)e^{zx}\,dz, \qquad x > 0,$$

$$K(x,y) = \frac{1}{2\pi i}\int_{L^{+}_{\delta}} M(z,y)e^{zx}\,dz, \qquad x < 0.$$

The inequality

$$\|K(x,y)\| < \frac{Ce^{-\delta|x|}}{|x|+\tau_1 y(1-y)}, \qquad 0 \le y \le 1$$

holds.

Proof. Let, for definiteness, $x < 0$. From Lemma 1 we obtain

$$\|K(x,y)\| < \frac{1}{2\pi}\int_{L^{+}_{\delta}} |e^{zx}|\,\|M(z,y)\|\,|dz| < C_1\int_0^{\infty} e^{-\delta|x|-r|x|\sin\eta_1}e^{-r\tau y(1-y)}\,dr =$$

$$= \frac{C_1e^{-\delta|x|}}{\tau y(1-y)+|x|\sin\eta_1} = \frac{Ce^{-\delta|x|}}{|x|+\tau_1 y(1-y)},$$

as was required.

We define the desired solution $u_0(x, y)$ by the formula

$$u_0(x,y) = \int_{-\infty}^{\infty} K(x-t,y)(f_1(t)-f_1(x))\,dt + \sum_{\operatorname{Re} z_n=0}\int_{-\infty}^{x} e^{z_n(x-t)}H_n(x-t,y)\times$$

$$\times f_1(t)\,dt + \frac{1}{2\pi i}\int_{L_0}\frac{M(z,y)}{z}\,dz\, f_1(x) = u_1(x,y)+u_2(x,y)+u_3(x,y),$$

where $f_1(x) = f(x)$ for $x \ge 0$, $f_1(x) = 0$ for $x \le -1$, and it satisfies condition (4).

Lemma 4. The function $u_0(x, y)$ is continuous for $x \ge 0$, $0 \le y \le 1$ and satisfies system (1). Moreover, $u_1(x, y) = O(\varphi(x))$.

Proof. We have

$$\|u_1(x,y)\| = \left\|\int_0^{\infty} K(u,y)(f_1(x-u)-f_1(x))\,du + \int_0^{\infty} K(-u,y)(f_1(x+u)-f_1(x))\,du\right\|$$

$$< C\int_0^\infty u^{\alpha-1}e^{-\delta u}\left[\varphi(x-u)+2\varphi(x)+\varphi(x+u)\right]du <$$

$$< C_1\varphi(x)\int_0^\infty u^{\alpha-1}e^{-(\delta-\delta_1)u}\,du = O(\varphi(x)),$$

since from $\lim\limits_{x\to+\infty}\dfrac{\varphi'(x)}{\varphi(x)} = 0$ it follows that, for any $\delta_1 > 0$,

$$\frac{\varphi(x+u)}{\varphi(x)} < C_{\delta_1}e^{\delta_1|u|}.$$

Hence we also obtain the continuity of u_1. The continuity of u_2 and u_3 is obvious.

To verify the boundary conditions, note that for $x \neq t$

$$B_1K(x-t,0) + B_2K(x-t,1) = 0$$

Thus, $B_1u_1(x,0) + B_2u_1(x,1) = 0$. Further, by Lemma 2,

$$B_1u_2(x,0) + B_2u_2(x,1) = 0,$$

$$B_1u_3(x,0)+B_2u_3(x,1) = \operatorname*{res}_{z=0}\frac{B_1M(z,0)+B_2M(z,1)}{z}f(x) = \operatorname*{res}_{z=0}\frac{E}{z}f(z) = f(x).$$

It remains to verify that for $x > 0,\ 0 < y < 1$ the derivatives $\partial u_0/\partial x,\ \partial u_0/\partial y$ exist and that $\partial u_0/\partial y = A\,\partial u_0/\partial x + Pu_0$. For $0 < y < 1$, in view of the estimates of Lemma 3, we have

$$\int_{-\infty}^{\infty}K(x-t,y)[f_1(t)-f_1(x)]\,dt = \int_{-\infty}^{\infty}K(x-t,y)f_1(t)\,dt - \int_{-\infty}^{\infty}K(u,y)\,du\,f_1(x).$$

Since $M(z,y) = O(e^{-\tau y(1-y)|z|})$, in the definition of $K(x,y)$ for $0 < y < 1$ integration over the contours L_δ^+ and L_δ^- may be replaced by integration along the straight lines $\operatorname{Re} z = \delta$ and $\operatorname{Re} z = -\delta$. Therefore

$$\int_{-\infty}^{\infty}K(u,y)\,du = \frac{1}{2\pi i}\left\{\int_{-\infty}^{0}\int_{\delta-i\infty}^{\delta+i\infty}M(z,y)e^{zu}\,dz\,du + \int_0^\infty\int_{-\delta-i\infty}^{-\delta+i\infty}M(z,y)e^{zu}\,dz\,du\right\} =$$

$$= \frac{1}{2\pi i}\left\{\int_{\delta-i\infty}^{\delta+i\infty}M(z,y)\frac{dz}{z} - \int_{-\delta-i\infty}^{-\delta+i\infty}M(z,y)\frac{dz}{z}\right\} = \frac{1}{2\pi i}\int_{L_0}M(z,y)\frac{dz}{z}.$$

Thus,

$$u_0(x,y) = \int_{-\infty}^{\infty} K(x-t,y) f_1(t)\,dt + \sum_{\operatorname{Re} z_n = 0} \int_{-\infty}^{x} e^{z_n(x-t)} H_n(x-t,y) f_1(t)\,dt =$$

$$= \int_{-\infty}^{\infty} K_1(x-t,y) f_1(t)\,dt,$$

where

$$K_1(x,y) = \frac{1}{2\pi i} \int_{\delta - i\infty}^{\delta + i\infty} M(z,y) e^{zx}\,dz, \qquad 0 < y < 1.$$

As in Lemma 3, we easily obtain the inequalities

$$\|K_1(x,y)\| < C_y e^{\delta x}, \qquad \left\| \frac{\partial}{\partial x} K_1(x,y) \right\| < C_y e^{\delta x}, \qquad x < 0.$$

From these inequalities and from 4) it is clear that differentiation under the integral sign is admissible, and the lemma is completely proved.

Applying Lemma 4 to each of the cases, we obtain Theorems 1-3.

The following example shows that in Theorems 1-3 one cannot make do with conditions only on the growth of $\|f(x)\|$. We take the Cauchy-Riemann system with boundary conditions

$$B_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad B_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and construct a bounded and differentiable $f(x)$ such that nothing can be said about the growth of the particular solution.

Namely, put

$$f(x) = \begin{pmatrix} \alpha(x,0) \\ \alpha(x,1) \end{pmatrix}, \qquad \text{where } \alpha(x,y) = \operatorname{Re} g(x+iy),$$

$$g(z) = i \sum_{n=1}^{\infty} \frac{1}{(e^{\pi z} - n)^2 + 1} \ln \frac{1 - i p_n (e^{\pi z} - n)}{p_n + 1}, \qquad p_n \to \infty.$$

The solution of least growth will be

$$u_0(x,y) = \begin{pmatrix} \operatorname{Re} g(x+iy) \\ \operatorname{Im} g(x+iy) \end{pmatrix},$$

and it is not difficult to check that, by choosing p_n, the function $u_0(x,y)$ can be made to tend to infinity arbitrarily fast.

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REFERENCES

1. M. A. Evgrafov, *DAN*, **126**, No. 3 (1959).

Note: Figure translations are in progress. See original paper for figures.