



Soviet-era science, translated into English

A. S. PEKELIS

1960

SovietRxiv

View the original and related papers at <https://sovietrxiv.org/items/ru-196001.53362>

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.

Abstract

Full Text

A. S. PEKELIS

ON STRUCTURAL ISOMORPHISMS OF SOLVABLE GROUPS

(Presented by Academician A. I. Mal'cev on 23 III 1960)

In G. Birkhoff's book ⁽¹⁾ the problem was formulated: is the solvability of a group preserved under structural isomorphisms? Suzuki gave a positive solution of this question for finite groups, and also gave a structural characterization of finite solvable groups ⁽²⁾. In the work ⁽³⁾, A. I. Mal'cev singled out classes of infinite solvable groups with certain finiteness conditions—solvable A_i -groups ($i = 1, 2, 3, 4, 5$). It is natural to check, for these classes of groups, whether the property of solvability is preserved under structural isomorphisms. In ^(4, 5) it was proved that for solvable A_i -groups ($i = 5, 4, 3$) this question is answered positively. For radical groups of finite special rank the corresponding question also has a positive solution ⁽⁴⁾. At the same time, the indicated classes of groups possess a structural characterization ⁽⁵⁾.

In the present note some classes of infinite groups are also considered for which the property of solvability or radicality is preserved under structural isomorphisms. From Theorem 1 it follows that the property of a group to be a nonperiodic solvable A_1 -group is a structural property. At the same time it is not difficult to give a structural characterization of nonperiodic solvable A_i -groups, where $i = 1, 2$.

§ 1. By φ we shall denote a structural isomorphism of two groups G and G^φ . If H is a subgroup in G , then by H^φ we shall denote its image in G^φ under the structural isomorphism φ .

We recall some known definitions. An element a of a structure S is called **Dedekindian** if for every pair of elements x, y of S , where $a \leq y$, the relation $(a+x)y = a+xy$ is satisfied. An element of a structure S is called **characteristic** if it is fixed under all automorphisms of the structure S . A subgroup H of a group G is called **S -characteristic** in G if H is a characteristic element in the subgroup structure $S(G)$ of the group G . It is clear that under structural isomorphisms S -characteristic subgroups correspond to S -characteristic subgroups. We note that every S -characteristic subgroup in a group G is its characteristic subgroup.

Lemma 1. Let $G = \{a, F\}$, where a is an element of infinite order, F is an S -characteristic periodic subgroup in G , and let H be a subgroup of F . Then H is a normal divisor in G if and only if H is a Dedekindian element in the subgroup structure $S(G)$ of the group G .

Proof. Sufficiency. From the property of Dedekindianity of H in $S(G)$ it follows that:

$$\{H, (af)H(af)^{-1}\} = \{H, af\} \cap \{H, (af)H(af)^{-1}\} = \{H, \{af\} \cap \{H, (af)H(af)^{-1}\}\},$$

where f is an arbitrary element of F . Since the element af is an element of infinite order, while the subgroup $\{H, (af)H(af)^{-1}\}$, contained in F , is periodic, it follows that

$$\{af\} \cap \{H, (af)H(af)^{-1}\} = e.$$

Hence we obtain that

$$\{H, (af)H(af)^{-1}\} = H.$$

Consequently, H is invariant with respect to every element af , where f is any element of F . In particular, H is invariant also with respect to the element a . Thus, H is invariant also with respect to any $f \in F$, i.e. H is a normal divisor in G .

Necessity is obvious.

Corollary. Let $G = \{a, P(G)\}$, where $P(G)$ is the maximal periodic normal divisor of the group G and a is an element of infinite order. If a subgroup H of $P(G)$ is invariant in G , then H^φ is a normal divisor in G^φ .

This assertion follows directly from the lemma, if one takes into account that $P(G)$ is an S -characteristic subgroup in G ⁽⁴⁾.

Theorem 1. *If G is a nonperiodic solvable A_1 -group, then G^φ is also a nonperiodic solvable A_1 -group.*

Proof. Let $P(G)$ be the maximal periodic normal divisor of the group G , and let

$$P(G) = K_0 \supset K_1 \supset \dots \supset K_{l-1} \supset K_l = e \quad (*)$$

be the commutator series of the subgroup $P(G)$. Since G is a nonperiodic group, by generating in G a subgroup with an element of infinite order and $P(G)$, we obtain, on the basis of the corollary to Lemma 1, that

$$P(G)^\varphi = K_0^\varphi \supset K_1^\varphi \supset \dots \supset K_{l-1}^\varphi \supset K_l^\varphi = e \quad (*)$$

is a series invariant in $P(G)^\varphi$, and $P(G)^\varphi = P(G^\varphi)$ is the maximal periodic normal divisor of the group G^φ ⁽⁴⁾. The structural isomorphism φ induces a structural isomorphism of the corresponding factor groups K_{i-1}/K_i and $K_{i-1}^\varphi/K_i^\varphi$ ($i = 1, \dots, l$). Therefore the factors of the series (*) are structurally isomorphic to periodic abelian groups. K. Iwasawa proved in ⁽⁷⁾ that a modular locally finite group is solvable, and moreover the number of members of its commutator series is at most two. Consequently, all factors of the series (*) are solvable, and hence $P(G^\varphi)$ is a solvable group. In this case the length of the commutator series of $P(G^\varphi)$ is not more than $2l$.

The structural isomorphism φ induces a structural isomorphism of the factor groups $G/P(G)$ and $G^\varphi/P(G^\varphi)$. The factor group $G/P(G)$ is a solvable A_4 -group ⁽³⁾. Therefore $G^\varphi/P(G^\varphi)$ is also a solvable A_4 -group ⁽⁴⁾. Hence the solvability of the group G^φ follows, and it is clear that G^φ will be a solvable A_1 -group. The theorem is proved.

Since a solvable A_2 -group can be characterized as a solvable A_1 -group of finite special rank, on the basis of Theorem 1 we obtain the following assertion:

Corollary. If G is a nonperiodic solvable A_2 -group, then G^φ is a nonperiodic solvable A_2 -group.

Remark. We note that, by Lemma 1, the solvability of a periodic S -characteristic subgroup H in a nonperiodic group G is characterized in a purely structural way: in H there exists a finite series of subgroups

$$e = H_0 \subset H_1 \subset \dots \subset H_l = H,$$

whose members are Dedekind elements in the structure of subgroups $S(G)$ of the group G , and all intervals H_i/H_{i-1} ($i = 1, \dots, l$) are modular. The maximal periodic normal divisor $P(G)$ of the group G is its S -characteristic subgroup ⁽⁴⁾. Since the factor group $G/P(G)$ of a solvable A_1 -group is a solvable A_4 -group ⁽³⁾, $G/P(G)$ is characterized purely structurally ⁽⁶⁾. Hence we obtain a structural characteristic of a nonperiodic solvable A_1 -group. Assuming that the group G has finite special rank, we obtain a structural characteristic of a nonperiodic solvable A_2 -group.

§ 2. In this paragraph the following assertion is proved:

Theorem 2. If G is a radical group and the series

$$e = H_0 \subset H_1 \subset \dots \subset H_\alpha \subset H_{\alpha+1} \subset \dots \subset H_\gamma = G$$

is its invariant series with locally nilpotent torsion-free factors, then G^φ is also a radical group, and

$$e = H_0^\varphi \subset H_1^\varphi \subset \dots \subset H_\alpha^\varphi \subset H_{\alpha+1}^\varphi \subset \dots \subset H_\gamma^\varphi = G^\varphi$$

is a series invariant in G^φ with locally nilpotent torsion-free factors.

The proof of this theorem is based on the following lemma:

Lemma 2. If H is an isolated locally nilpotent normal divisor in a torsion-free group G , then H^φ is an isolated locally nilpotent normal divisor in the group G^φ .*

Proof. The isolation and local nilpotence of the subgroup H^φ follow immediately from ⁽⁸⁾. In order to prove the invariance of the subgroup H^φ in G^φ , it is enough to assume that $G = \langle g, H \rangle$. Take in G the intersection F of all subgroups H^σ , where σ is any automorphism of the subgroup structure of the group G . Then F is an S -characteristic subgroup in G . We have

$$F = \bigcap_{\sigma \in A(S)} H^\sigma = \bigcap_{\sigma \in A(S)} (H^\sigma \cap H),$$

where $A(S)$ is the group of automorphisms of the subgroup structure of the group G . We shall show that if $H^\sigma \neq H$, then $H^\sigma \cap H$ is a maximal isolated subgroup in H . We do this analogously to the proof of a similar assertion in Theorem 5.1 of ⁽⁴⁾. Denote $D = H^\sigma \cap H$. Since G/H is infinite cyclic, D is maximal isolated in H^σ . Then $D^{\sigma^{-1}} = H \cap H^{\sigma^{-1}}$ is a maximal isolated subgroup in H (σ^{-1} is the inverse structural isomorphism). Consequently, the series $D^{\sigma^{-1}} \subset H \subset G$ is a rational series of length 2 ⁽⁹⁾. Hence, between $D^{\sigma^{-1}}$ and G one can insert only one subgroup in order to obtain a dense series ⁽⁹⁾. The analogous assertion is true for the subgroup D and the group G . But D is isolated in H ; hence D is a maximal isolated subgroup in H and, consequently, invariant in H ⁽⁹⁾.

F contains the intersection of the maximal isolated subgroups of H . Therefore H/F is a torsion-free abelian group. The structural isomorphism φ induces a structural isomorphism of the factor groups G/F and G^φ/F^φ . H/F is an isolated abelian normal divisor in G/F . Then H^φ/F^φ is an isolated abelian normal divisor in G^φ/F^φ ⁽¹⁰⁾. Hence the invariance of H^φ in G^φ follows. The lemma is proved.

Corollary. If, in a torsion-free group G , the radical $R(G)$ is isolated, then $R(G)$ is an S -characteristic subgroup in G .

Remark. Let G be a radical group all members of whose radical series are locally nilpotent torsion-free groups. Then, from the corollary to Lemma 2, we obtain that all members of this series are S -characteristic subgroups of the group G . And since a locally nilpotent torsion-free group is characterized purely structurally ⁽⁸⁾, it is therefore easy to obtain a structural characteristic of the indicated group.

The author expresses gratitude to Prof. B. I. Plotkin for his attention to this work.

Sverdlovsk State
Pedagogical Institute

Received
14 III 1960

REFERENCES CITED

1. G. Birkhoff, *Lattice Theory*, IL, 1952.
2. M. Suzuki, "Structure of Group and the Structure of its Lattice of Subgroups," *Ergebn. Math.*, No. 10 (1956).

3. A. I. Mal' tsev, *Matem. sborn.*, **28** (70), 3, 567 (1951).
4. B. I. Plotkin, *Tr. Mosk. matem. obshch.*, **6**, 299 (1957).
5. A. S. Pekelis, *Uch. zap. Ural' sk. gos. univ.*, ser. matem., **19**, 51 (1956).
6. A. S. Pekelis, *UMN*, **13**, issue 3, 238 (1958).
7. K. Iwasawa, *Japan J. Math.*, **18**, 709 (1943).
8. P. G. Kontorovich, B. I. Plotkin, *Matem. sborn.*, **35** (77), 1, 187 (1954).
9. B. I. Plotkin, *Matem. sborn.*, **30** (72), 1, 197 (1952).
10. A. S. Pekelis, *UMN*, **11**, issue 4, 143 (1956).

* An analogous assertion for R -groups was proved by B. I. Plotkin in ⁽⁴⁾.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.