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Abstract

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MATHEMATICS

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ON A CERTAIN CLASS OF KAWAGUCHI SPACES

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A **Kawaguchi space** is a space X_n in which an integral is given

$$\int_{t_1}^{t_2} L\left(\xi^{\alpha}, \xi^{(i)\alpha}\right) dt \qquad (\alpha, \beta, \ldots = 1, 2, \ldots, n;\ r, j = 1, 2, \ldots, \nu), \tag{1}$$

where $\xi^{(i)\alpha} = d^i\xi^{\alpha}/dt^i$, the value of which for any arc of an oriented curve is called the **length of the arc** of this curve. In order that the value of the integral (1) not depend on the choice of the parameter t of the curve, as is known ([5]), it is necessary and sufficient that the function $L\left(\xi^{\alpha}, \xi^{(i)\alpha}\right)$ satisfy the conditions

$$\sum_{k=i}^{\nu} C_k^i \frac{\partial L}{\partial \xi^{(k)\alpha}}\, \xi^{(k-i+1)\alpha} = \delta_i^1 L \qquad (i = 1, 2, \ldots, \nu), \tag{2}$$

where δ_j^i is the Kronecker symbol. It is known that every Klein space has an integral invariant defining the invariant length of arcs of curves, and, consequently, can be considered as a Kawaguchi space.

The theory of Kawaguchi spaces of general type, constructed by A. Kawaguchi and many others, mainly Japanese geometers, is very complicated; in particular because the base space must be taken to be a manifold of linear elements of high order. This circumstance also makes it difficult to compare Kawaguchi spaces with known geometries. The class of Kawaguchi spaces singled out by us is of interest already because in it an affine connection is constructed, while the study of motions in these spaces reveals their connection with equiaffine and symplectic geometries.

Let us consider Kawaguchi spaces for which

$$L\left(\xi^{\alpha},\xi^{(i)\alpha}\right)=\left[\sum_{k=\nu}^{N}\sum_{\substack{i_1+i_2+\cdots+i_k=N\\ \nu\geq i_1\geq i_2\geq\cdots\geq i_k}} a^{i_1 i_2\ldots i_k}_{\alpha_1\alpha_2\ldots\alpha_k}\,\xi^{(i_1)\alpha_1}\xi^{(i_2)\alpha_2}\ldots\xi^{(i_k)\alpha_k}\right]^{1/N}, \qquad (3)$$

where $N=\nu(\nu+1)/2$, the $a^{i_1 i_2\ldots i_k}_{\alpha_1\alpha_2\ldots\alpha_k}$ are functions of ξ^{α}, differentiable a sufficient number of times, and symmetric in the lower indices to which identical upper indices correspond; moreover, among the $a^{i_1 i_2\ldots i_\nu}_{\alpha_1\alpha_2\ldots\alpha_\nu}$ only

$$a^{\nu\ \nu-1\ldots 1}_{\alpha_1\alpha_2\ldots\alpha_\nu}$$

are different from zero. It is easy to see that the quantities

$$a^{i_1 i_2\ldots i_k}_{\alpha_1\alpha_2\ldots\alpha_k}$$

are components of a certain linear geometric object of class ν. Conditions (2) impose certain restrictions on

this object; in particular, it is not difficult to verify that $a_{\alpha_1\alpha_2\ldots\alpha_\nu}$ are skew-symmetric in the lower indices. The quantities $a_{\alpha_1\alpha_2\ldots\alpha_\nu}$ define a ν-vector and, together with $a^{\nu}_{\alpha_1}\,a^{\nu-1}_{\alpha_2}\ldots a^{3}_{\alpha_{\nu-2}}a^{1}_{\alpha_{\nu-1}}\,a^{1}_{\alpha_\nu}\,a^{1}_{\alpha_{\nu+1}}$, define a geometric object of the second class.

Suppose that from the components of the ν-vector $a_{\alpha_1\alpha_2\ldots\alpha_\nu}$ one can construct a nonzero invariant $\mathfrak{A}$ of weight w. This invariant may always be regarded as a homogeneous polynomial, with respect to the components $a_{\alpha_1\alpha_2\ldots\alpha_\nu}$, of degree ρ, where $\rho\nu=nw$. It should be noted that, for fixed ν, this condition may fail to hold for many values of n, which must be excluded from consideration; thus, for example, when $\nu=2$, the given condition can hold only for even n. The Kawaguchi spaces of type (3) satisfying this condition will be denoted by L^{ν}_{n}. An example of the indicated invariant is $A_{[12\ldots n][12\ldots n]\ldots[12\ldots n]}$, where $A_{\alpha_1\alpha_2\ldots\alpha_r}$ is the product of $a_{\alpha_1\alpha_2\ldots\alpha_\nu}$ by itself m times, with $\nu m=nl=r$, where r is the least common multiple of n and ν. This invariant is, generally speaking, nonzero when $\nu=2$ and n is even, and when $\nu=n$.

It is known ([6]) that the contravariant ν-vector $a^{\alpha_1\alpha_2\ldots\alpha_\nu}=\dfrac{w}{\nu\mathfrak{A}}\dfrac{\partial\mathfrak{A}}{\partial a_{\alpha_1\alpha_2\ldots\alpha_\nu}}$ satisfies the relation

$$a_{\alpha_1\alpha_2\ldots\alpha_{\nu-1}\alpha}a^{\alpha_1\alpha_2\ldots\alpha_{\nu-1}\beta}=\delta^{\beta}_{\alpha}.$$

It can be shown that the quantities

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{\nu+1} a^{\alpha_1\alpha_2\dots\alpha_{\nu-1}\alpha} \left(2\partial_{(\beta} a_{\alpha_1\alpha_2\dots\alpha_{\nu-1}|\gamma)} - 3(\nu-1) a^{\nu}_{\alpha_1} a^{\nu-1}_{\alpha_2} \dots a^{3}_{\alpha_{\nu-2}} a^{1}_{\nu-1} a^{1}_{\beta} a^{1}_{\gamma} \right), \tag{4}$$

where ∂_β is the symbol of partial differentiation with respect to ξ^β, define an object of symmetric affine connection. In the case $n = \nu = 2$, such a connection is mentioned by É. Cartan ([1]), and for $\nu = 2$ and arbitrary even n it can be obtained from the connection defined on the set of linear elements of the first order, constructed by A. Kawaguchi for more general Kawaguchi spaces of the second order ([2]). Recently the spaces L^2_n have been studied from another point of view by V. G. Lemlein ([3]) and Yu. I. Levin ([4]).

For the connection (4), when $\nu = 2$, the equality

$$\nabla_\gamma a_{\alpha\beta} = \partial_{[\gamma} a_{\alpha\beta]},$$

holds, and when $\nu = n$, the equality

$$\nabla_\alpha a_{\alpha_1\alpha_2\dots\alpha_\nu} = 0.$$

We shall call a **conformal transformation** of a Kawaguchi space a transformation under which its metric function changes as follows:

$$\bar{L}(\xi^\alpha, \xi^{(i)\alpha}) = \sigma(\xi^\alpha) L(\xi^\alpha, \xi^{(i)\alpha}).$$

Under a conformal transformation, the connection (4) and the vector $S_\beta = \nabla_\beta \mathfrak{A}/\mathfrak{A}$ transform as follows:

$$\bar{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} + \sigma_{(\beta}\delta^{\alpha}_{\gamma)}, \qquad \bar{S}_\beta = S_\beta + \frac{\rho - w}{2}\sigma_\beta \quad \left(\sigma_\beta = \frac{2}{\nu+1} \frac{\partial_\beta \sigma}{\sigma} \right).$$

Consequently, when $\nu \neq n$, the new object of symmetric affine connection

$$\tilde{\Gamma}^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\beta\gamma} - \frac{2}{\rho - w} S_{(\beta}\delta^{\alpha}_{\gamma)} \tag{5}$$

is invariant under conformal transformations.

Expressing $\xi^{(i)\alpha}$ in terms of the absolute derivatives $\delta^i\xi^\alpha/dt^i$, found with the aid of the connection (4) or (5), and substituting their expressions into (3), we obtain

$$L(\xi^\alpha, \xi^{(i)\alpha}) = \left[\sum_{k=\nu}^{N} \sum_{\substack{i_1+i_2+\dots+i_k=N \\ \nu \geq i_1 \geq i_2 \geq \dots \geq i_k}} b^{i_1\,i_2\dots i_k}_{\alpha_1\alpha_2\dots\alpha_k} \frac{\delta^{i_1}\xi^{\alpha_1}}{dt^{i_1}} \frac{\delta^{i_2}\xi^{\alpha_2}}{dt^{i_2}} \cdots \frac{\delta^{i_k}\xi^{\alpha_k}}{dt^{i_k}} \right]^{1/N} . \tag{6}$$

The quantities $b^{i_1\,i_2\dots i_k}_{\alpha_1\alpha_2\dots\alpha_k}$ are determined by this equality, obviously, uniquely and, for a fixed sequence $i_1, i_2, \dots, i_k$, are components of a covariant tensor of valence k, with $b^{\nu\,\nu-1\dots1}_{\alpha_1\alpha_2\dots\alpha_\nu} = a_{\alpha_1\alpha_2\dots\alpha_\nu}$. In the spaces L^2_n, if the connection (4) is used, the function $L(\xi^\alpha, \xi^{(i)\alpha})$ takes the following simple form:

$$L(\xi^\alpha, \xi^{(1)\alpha}, \xi^{(2)\alpha}) = \left[a_{\alpha\beta} \frac{\delta^2\xi^\alpha}{dt^2} \frac{\delta\xi^\beta}{dt} \right]^{1/3} .$$

Thus we obtain:

Theorem 1. *The spaces L^ν_n are spaces of symmetric affine connection with a prescribed system of covariant tensors of valences $\nu, \nu+1, \dots, N$.*

Let us consider the question of the existence of motions in the given spaces. A vector field v^α determines a one-parameter group of transformations of L^ν_n, with respect to which the connection (4) and the ν-vector $a_{\alpha_1\alpha_2\dots\alpha_\nu}$ are invariant if and only if v^α satisfies the system of equations

$$\begin{aligned} \nabla_\beta v^\alpha &= v^\alpha_\beta, \\ \nabla_\gamma v^\alpha_\beta &= -R^\alpha_{\lambda\gamma\beta} v^\lambda, \\ v^\lambda \nabla_\lambda a_{\alpha_1\alpha_2\dots\alpha_\nu} + \nu a_{\lambda[\alpha_2\dots\alpha_\nu} v^\lambda{}_{\alpha_1]} &= 0, \end{aligned} \tag{7}$$

where for covariant differentiation the connection (4) is used, and $R^\lambda_{\alpha\beta\gamma}$ is the curvature tensor of this connection. The conditions for complete integrability of the system (7) have the form

$$\begin{aligned} 2\nabla_{[\mu} R^\alpha_{\gamma]\lambda\beta} &= -\nabla_\lambda a_{\alpha_1\alpha_2\dots\alpha_\nu} K^{\alpha_1\alpha_2\dots\alpha_\nu\alpha}_{\mu\gamma\beta}, \\ R^\omega_{\mu\gamma\beta}\delta^\alpha_\lambda - R^\alpha_{\mu\gamma\lambda}\delta^\omega_\beta - 2R^\alpha_{\lambda[\gamma|\beta|}\delta^\omega_{\mu]} &= \nu a_{\lambda[\alpha_2\dots\alpha_\nu}\delta^\omega_{\alpha_1]} K^{\alpha_1\alpha_2\dots\alpha_\nu\alpha}_{\mu\gamma\beta}, \\ \nabla_\mu\nabla_\lambda a_{\alpha_1\alpha_2\dots\alpha_\nu} + \nu R^\omega_{\mu\lambda[\alpha_1} a_{|\omega|\alpha_2\dots\alpha_\nu]} &= \nabla_\lambda a_{\beta_1\beta_2\dots\beta_\nu} K^{\beta_1\beta_2\dots\beta_\nu}_{\alpha_1\alpha_2\dots\alpha_\nu\mu}, \\ \nabla_\lambda a_{\alpha_1\alpha_2\dots\alpha_\nu}\delta^\omega_\mu + \nu\nabla_\mu a_{\lambda[\alpha_2\dots\alpha_\nu}\delta^\omega_{\alpha_1]} &= \nu a_{\lambda[\beta_2\dots\beta_\nu}\delta^\omega_{\beta_1]} K^{\beta_1\beta_2\dots\beta_\nu}_{\alpha_1\alpha_2\dots\alpha_\nu\mu}, \end{aligned} \tag{8}$$

where $K^{\alpha_1\alpha_2\dots\alpha_\nu\alpha}_{\mu\gamma\beta}$ and $K^{\beta_1\beta_2\dots\beta_\nu}_{\alpha_1\alpha_2\dots\alpha_\nu\mu}$ are certain tensors. We shall restrict ourselves to considering the spaces L^2_n and L^n_n. A rather cumbersome investigation of the conditions (8) shows that for L^2_n they are equivalent to the conditions

$$\nabla_\gamma a_{\alpha\beta} = 0, \qquad R^\lambda_{\alpha\beta\gamma} = 0, \tag{9}$$

and for L_n^n to the condition

$$R^\lambda_{\alpha\beta\gamma} = 0. \tag{10}$$

Thus, in order that the space L_n^2 have a group of motions depending on the maximal number of parameters, which, as is easily seen from the system (7), is equal to $n(n+3)/2$, it is necessary and sufficient that the conditions (9) be satisfied.

In the case of the space L_n^n we require that all tensors $b^{i_1 i_2 \dots i_k}_{\alpha_1 \alpha_2 \dots \alpha_k}$ $(n < k \leq N)$ be invariant with respect to the group of transformations defined by the system (7). For this it is necessary and sufficient that the conditions

$$v^\lambda \nabla_\lambda b^{i_1 i_2 \dots i_k}_{\alpha_1 \alpha_2 \dots \alpha_k} + \sum_{l=1}^{k} b^{i_1 i_2 \dots i_{l-1} i_l i_{l+1} \dots i_k}_{\alpha_1 \alpha_2 \dots \alpha_{l-1} \lambda \alpha_{l+1} \dots \alpha_k} \partial_{\alpha_l} v^\lambda = 0$$

be consequences of the last equality (7). But it follows from this that any tensor $b^{i_1 i_2 \dots i_k}_{\alpha_1 \alpha_2 \dots \alpha_k}$ has one essential component, which is a density of some integral weight. Then one can represent the scalar

$$b^{i_1 i_2 \dots i_k}_{\alpha_1 \alpha_2 \dots \alpha_k} \frac{\delta^{i_1} \xi^{\alpha_1}}{dt^{i_1}} \frac{\delta^{i_2} \xi^{\alpha_2}}{dt^{i_2}} \cdots \frac{\delta^{i_k} \xi^{\alpha_k}}{dt^{i_k}}$$

as the product of the given density by a density constructed from the vectors

$$\frac{\delta \xi^\alpha}{dt}, \quad \frac{\delta^2 \xi^\alpha}{dt^2}, \dots, \quad \frac{\delta^n \xi^\alpha}{dt^n},$$

and this, in view of the condition $i_1 + i_2 + \dots + i_k = N$, is possible only when $b^{i_1 i_2 \dots i_k}_{\alpha_1 \alpha_2 \dots \alpha_k} = 0$.

Thus, the space L_n^n has a group of motions depending on the maximal number of parameters, which, as follows from the system (7), is equal to $n(n+1) - 1$, if and only if condition (10) is satisfied and all tensors $b^{i_1 i_2 \dots i_k}_{\alpha_1 \alpha_2 \dots \alpha_k}$ for $n < k \leq N$ are equal to zero.

From conditions (9) and (10) there follows the existence of coordinate systems in which $\Gamma^\alpha_{\beta\gamma} = 0$ and $a_{\alpha_1 \alpha_2 \dots \alpha_\nu}$ does not depend on the point of the space.

Finally we obtain:

Theorem 2. *A space L_n^2 possessing a group of motions depending on the maximal number $n(n+3)/2$ of parameters is a symplectic space.*

Theorem 3. *A space L_n^n possessing a group of motions depending on the maximal number $n(n+1)-1$ of parameters is an equiaffine space.*

These theorems show, in particular, that equiaffine and symplectic spaces are completely determined by their structure as Kawaguchi spaces, which is defined by the integral invariant of the least differential order of these spaces.

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Note: Figure translations are in progress. See original paper for figures.