# ON EXTERNAL BASES OF SETS LYING IN BICOMPACTA

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## Abstract

## Full Text

## MATHEMATICS

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# ON EXTERNAL BASES OF SETS LYING IN BICOMPACTA

*(Presented by Academician P. S. Aleksandrov, 22 I 1960)*

In all that follows the spaces $X$ and $Y$ will be subspaces of some fixed space $R$, and the terms "open," "closed" will mean "open in $R$," respectively "closed in $R$."

**Definition 1.** A system of sets $\gamma = \{E_\alpha : \alpha \in A\}$ of the space $R$ is called a **separating system of the pair of spaces** $(X, Y)$ if, for every pair of points $x \in X$, $y \in Y$, there are $E_\alpha \in \gamma$, $E_\beta \in \gamma$ such that $E_\alpha \ni x$, $E_\beta \ni y$, and $E_\alpha \cap E_\beta = \Lambda$.

If $R$ is normal, then, as is easy to see, the following holds.

**Lemma 1.** Let $\gamma = \{E_\alpha : \alpha \in A\}$ be a separating system of the pair $(X, Y)$ consisting of closed sets. Then there exists a system $\tilde{\gamma}$ of open sets, separating the pair $(X, Y)$ and having the same cardinality as the system $\gamma$.

For the proof of Lemma 1 it is enough to take, for each pair of disjoint sets $E_\alpha, E_\beta$, $\alpha, \beta \in A$, disjoint neighborhoods of them in the space $R$: $O_\alpha^\beta$ and $O_\beta^\alpha$. By normality of the space $R$, this can always be done. The totality of the open sets selected forms the required system $\tilde{\gamma}$.

**Definition 2.** Let $(X, Y)$ be an arbitrary pair of subspaces of some containing space $R$. We shall call a system $B_Y^X$ of open sets of the space $R$ a **base of the space** $X$ **relative to the space** $Y$ if, for every point $x \in X$ and every one of its neighborhoods $Ox$ (in the space $R$), there is a $\Gamma_x \in B_Y^X$ such that $x \in \Gamma_x$ and $\Gamma_x \cap Y \subseteq Ox$. A special case: $R = X \cup Y$. Still more special cases are:

a) $Y = X$; then the base $B_Y^X$ is a base of the space $X$;

b) $X \subset Y$; then $B_Y^X$ is an external base of the space $X$ in the space $Y$;

c) $X \cap [Y] = \Lambda$; then as $B_Y^X$ one may take a system consisting of the single set $X$.

**Lemma 2.** Let $\gamma = \{E_\alpha : \alpha \in A\}$ be a system of open sets separating the pair $(X, Y)$, and suppose $Y$ is bicompact. Then there exists a base of the space $X$ relative to the space $Y$ of the same cardinality as the system $\gamma$.

**Proof.** Consider the system $\tilde{\gamma}$ consisting of all possible finite intersections of sets of the system $\gamma$. We shall show that $\tilde{\gamma}$ forms a base of the space $X$ relative to the space $Y$. Let $x_1$ be an arbitrary point of the space $X$, and let $Ox_1 \ni x_1$ be an arbitrary neighborhood of it (in $R$). Then $\Phi_1 = Y \setminus Ox_1$ is a closed subset of $Y$ and therefore is bicompact. For each point $y \in \Phi_1$ there is a pair of sets $A_y, C_y$ such that $A_y \ni x_1$, $C_y \ni y$, $A_y \cap C_y = \Lambda$, and $A_y \in \gamma$, $C_y \in \gamma$.

The collection of sets $C_y$, where $y$ ranges over $\Phi_1$, forms a cover of the set $\Phi_1$, and, by bicompactness of $\Phi_1$, from this cover one can choose···

a finite subcover $\{C_{y_1}, \dots, C_{y_k}\}$. Then for the set

$$A_{x_1} = \bigcap_{i=1}^{k} A_{y_i}$$

the following conditions are satisfied: $A_{x_1} \in \tilde{\gamma}$; $A_{x_1} \ni x_1$; $A_{x_1} \cap Y \subset Ox_1$ (the last holds by virtue of the relation $(A_{x_1}) \cap Y \setminus Ox_1 = A_{x_1} \cap \Phi_1 = \Lambda$). Lemma 2 is proved.

**Lemma 3.** *Let $X \subseteq Y = R$; $X$ be a Borel set in $Y$ of type $G_{(\tau)}$. Then the cardinality of the system $\Sigma = \{G\}$ of all open sets of the space $Y$ participating in the given representation of the set $X$ does not exceed $\tau$, and for arbitrary points $x \in X$, $y \in Y \setminus X$ there is a $G \in \Sigma$ such that $x \in G$, $y \notin G$.*

Both assertions of the lemma are easily proved by induction (on the class of the representation of the set $X$). We confine ourselves to the proof of the second assertion. Let $X$ be of class $\lambda$. Two cases are possible: a) $X = \bigcup_s X_\alpha$; b) $X = \bigcap_\alpha X_\alpha$, where the $X_\alpha$ are sets given by representations of classes $< \lambda$. Then there exists an $\alpha$ such that $x \in X_\alpha$, $y \in R \setminus X_\alpha$, and, since $X_\alpha$ is represented by operations over all or some $G \in \Sigma$, but already by class $< \lambda$, there exists, by the induction hypothesis, such a $G \in \Sigma$ that $x \in G$, $y \notin G$.

On the basis of the three lemmas proved above we easily obtain the proof of the main theorem of this note.

**Main theorem.** *Let $X \subseteq \Phi$, $\Phi$ be a bicompactum, and $X$ a Borel set of type $G_{(\tau)}$ in $\Phi$. Then from the existence of a network $*$ of the space $X$ of cardinality $\tau$ there follows the existence of an external base of the space $X$ in the space $\Phi$ having the same cardinality $\tau$.*

**Proof.** Consider $\Sigma = \{G^\alpha\}$—the system of open sets of the space $\Phi$ participating in the given representation of the set $X$, and put $F^\alpha = \Phi \setminus G^\alpha$. The cardinality of the system $\gamma_1 = \{F^\alpha\}$ does not exceed $\tau$ by Lemma 3. Moreover, by the same lemma, for any pair of points $x \in X$, $y \in Y \setminus X$ there is an $F^\alpha \ni y$, $F^\alpha \not\ni x$. Finally, all $F^\alpha$ are closed sets in $\Phi$ (and hence bicompacta). Denote by $\gamma_2 = \{P^\alpha\}$ the system consisting of the closures in $\Phi$ of the elements of a network of cardinality $\leq \tau$ of the space $X$. The cardinality of the system $\gamma_2$, obviously, does not exceed $\tau$.

The system $\gamma = \gamma_1 \cup \gamma_2$ consists of closed sets of the space $\Phi$; its cardinality does not exceed $\tau$; by the properties of the systems $\gamma_1$ and $\gamma_2$, the system $\gamma$ separates the pair $(X, \Phi)$. By Lemma 1 there exists a system of open sets of the space $\Phi$ of the same cardinality which also separates the pair $(X, \Phi)$. By Lemma 2 there exists a base of the space $X$ in the space $\Phi$ of the same cardinality, and by the relation $X \subseteq \Phi$ this base turns out to be an external base of the space $X$ in the space $\Phi$. The theorem is proved.

The theorem implies the following corollaries:

1. Under a continuous mapping onto a space of type $G_{(\aleph_0)}$ in a bicompactum, in particular onto a complete space in the sense of Čech, the weight cannot increase.
2. In order that a space $X$ of type $F_{(\aleph_0)}$, where the original sets $F$ are compacta (or, in general, arbitrary spaces with a countable base), be metrizable, it is sufficient that it be of type $G_{(\aleph_0)}$ in some bicompactum $\Phi \supseteq X$; moreover, then it itself has a countable base.

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## References

1. A. Arhangel' skii, DAN, **126**, No. 2, 239 (1959).

* By a network ([1]) of the space $X$ is meant such a system $\Sigma = \{A_\alpha\}$ of sets of this space that for any point $x \in X$ and any neighborhood $Ox$ of it there is an $A_\alpha \in \Sigma$ such that $x \in A_\alpha \subseteq Ox$.

*Note: Figure translations are in progress. See original paper for figures.*