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## Abstract

## Full Text

MATHEMATICS

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# COMPLEX ENVELOPES OF COMPACT HOMOGENEOUS SPACES

(Presented by Academician P. S. Aleksandrov on 16 IX 1959)

Let  $\mathfrak{K}$  be a connected compact Lie group,  $\Omega$  its connected closed subgroup, and  $\mathfrak{o}(X)$  the algebra of spherical functions on the homogeneous space  $X = \mathfrak{K}/\Omega$ . In the present note we study homomorphisms of the algebra  $\mathfrak{o}(X)$  into the field  $C$  of complex numbers. It is proved that these homomorphisms are in natural correspondence with the points of a certain complex manifold  $Z$ , which contains  $X$  and on which the complex envelope of the group  $\mathfrak{K}$  acts transitively. The manifold  $Z$  can be realized as an algebraic submanifold of a complex affine space. The results are applied to the study of inclusions between transitive complex transformation groups and to the study of homogeneous Stein manifolds.

1. Let  $\mathfrak{K}$  be a connected compact Lie group,  $K$  its Lie algebra,  $G = K^{C*}$ ,  $i : K \rightarrow G$  the natural embedding. Suppose that a connected complex Lie group  $\mathfrak{G}$  and a continuous monomorphism  $\alpha : \mathfrak{K} \rightarrow \mathfrak{G}$  are given, satisfying the following conditions: 1) the Lie algebra of the group  $\mathfrak{G}$  is  $G$ , and  $\dot{\alpha} = i$ ; 2)  $\alpha(\mathfrak{K})$  is a maximal compact subgroup of the group  $\mathfrak{G}$ . Then we say that  $\mathfrak{G}$  is the **complex envelope** of the group  $\mathfrak{K}$ .\*\* If the group  $\mathfrak{K}$  is semisimple, then condition 2) follows from 1).

**Lemma 1.** *Every connected compact Lie group  $\mathfrak{K}$  has a complex envelope. If  $\mathfrak{G}, \tilde{\mathfrak{G}}$  are two complex envelopes of the group  $\mathfrak{K}$ , and  $\alpha, \tilde{\alpha}$  the corresponding monomorphisms, then there exists a complex isomorphism  $\beta : \mathfrak{G} \rightarrow \tilde{\mathfrak{G}}$  such that  $\tilde{\alpha} = \beta\alpha$ .*

It follows from this that the complex envelope  $\mathfrak{G}$  of the group  $\mathfrak{K}$  is unique. We shall identify  $\mathfrak{K}$  with the subgroup  $\alpha(\mathfrak{K}) \subset \mathfrak{G}$ .

**Lemma 2.** *Let  $\mathfrak{K}, \Omega$  be connected compact Lie groups; let  $\mathfrak{G}, \mathfrak{H}$  be their complex envelopes. Every continuous homomorphism  $\varphi : \Omega \rightarrow \mathfrak{K}$  extends (and in a unique way) to a complex-analytic homomorphism  $\psi : \mathfrak{H} \rightarrow \mathfrak{G}$ . If the group  $\text{Ker } \varphi$  is finite, then  $\text{Ker } \psi = \text{Ker } \varphi$ .*

Let, in particular,  $\varphi$  be a continuous unitary representation of the group  $\Omega$ , i.e. a homomorphism of the group  $\Omega$  into  $U(n)$ . Obviously, the complex envelope of the group  $U(n)$  is the full linear group  $GL(n, C)$ . Therefore it follows from Lemma 2 that there exists a complex-analytic representation of the group  $\mathfrak{H}$  extending  $\varphi$ .

Let  $\mathfrak{G}$  be a complex envelope, and let  $\Omega$  be a connected closed subgroup of the group  $\mathfrak{K}$ . It follows from Lemma 2 that the complex envelope  $\mathfrak{H}$  of the group  $\Omega$  may be identified with a complex-analytic subgroup of the group  $\mathfrak{G}$ . It is not difficult to show that  $\mathfrak{H}$  is closed in  $\mathfrak{G}$ . Therefore the subgroup  $\mathfrak{H} \cap \mathfrak{K}$  is com-

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\* If  $R$  is a real vector space (or Lie algebra), then its complex envelope is denoted by  $R^C$ .

\*\* For another definition of the complex envelope, see <sup>(1)</sup>.

compact. Since it contains a maximal compact subgroup  $\Omega$  of the group  $\mathfrak{H}$ , we have  $\mathfrak{H} \cap \mathfrak{K} = \Omega$ . This thereby determines an embedding  $X \rightarrow Z$ , where  $X = \mathfrak{K}/\Omega$ ,  $Z = \mathfrak{G}/\mathfrak{H}$ . We shall say that the homogeneous space  $Z$  is a **complex envelope** of the homogeneous space  $X$ , and shall identify  $X$  with a submanifold of  $Z$ .

Let  $M = \mathfrak{G}/\mathfrak{H}$  be some homogeneous space, and let  $V$  be a finite-dimensional vector space over the field  $C$ . A mapping  $\Phi : M \rightarrow V$  is called **covariant** if there exists a continuous representation  $\varphi$  of the group  $\mathfrak{G}$  in the space  $V$  such that

$$\Phi(gx) = \varphi(g)\Phi(x) \quad (x \in M, g \in \mathfrak{G}).$$

A covariant mapping is always analytic; and if  $\mathfrak{G}$ ,  $M$ , and  $\varphi$  are complex-analytic, then  $\Phi$  is a complex-analytic mapping. A covariant mapping  $\Phi$  is called **real** if there exists a real subspace  $V_R \subset V$  such that  $V_R^C = V$  and  $\Phi(M) \subset V_R$ .

**Lemma 3.** *Let  $Z$  be the complex envelope of a compact homogeneous space  $X$ . Then every covariant mapping  $\Phi : X \rightarrow V$  extends (and moreover uniquely) to a complex-analytic covariant mapping  $\Psi : Z \rightarrow V$ . If  $\Phi$  is real and one-to-one, then  $\Psi$  is also one-to-one.*

Let us note that every homogeneous space  $\mathfrak{K}/\Omega$ , where  $\mathfrak{K}$  and  $\Omega$  are compact Lie groups, admits a real one-to-one covariant mapping into some space  $V$  <sup>(2)</sup>.

- Let  $M$  be a homogeneous space of the Lie group  $\mathfrak{G}$ . Denote by  $F(M)$  the algebra over the field  $C$  of all continuous complex-valued functions on  $M$ . To each element  $g \in \mathfrak{G}$  there naturally corresponds an automorphism  $g^*$  of the algebra  $F(M)$ . For each  $f \in F(M)$ , denote by  $R_f$  the linear subspace in  $F(M)$  spanned by functions of the form  $g^*f$  ( $g \in \mathfrak{G}$ ). A function  $f$  is called **spherical** if the space  $R_f$  is finite-dimensional. The set of all spherical functions is a subalgebra of the algebra  $F(M)$ . If  $M = \mathfrak{G}$ , then this subalgebra coincides with the algebra of representative functions on the group  $\mathfrak{G}$  <sup>(1)</sup>.

Let  $Z = \mathfrak{G}/\mathfrak{H}$  be the complex envelope of the compact homogeneous space  $X = \mathfrak{K}/\Omega$ . Denote by  $\mathfrak{o}(X)$  the algebra of spherical functions on  $X$ , and by  $\mathfrak{o}(Z)$  the algebra of complex-analytic spherical functions on  $Z$ . Let  $f \in \mathfrak{o}(X)$ . Consider the representation  $\varphi$  of the group  $\mathfrak{K}$  in the space  $R_f$ , acting by the formula  $\varphi(k)h = k^*h$  ( $k \in \mathfrak{K}$ ,  $h \in R_f$ ), and denote by  $\psi$  the representation

of the group  $\mathfrak{G}$  in  $R_f$  extending  $\varphi$ . If  $o \in X$  is the point corresponding to the subgroup  $\Omega$ , then set

$$(Pf)(go) = (\psi(g^{-1})f)(o) \quad (g \in \mathfrak{G}).$$

This defines a mapping  $P : \mathfrak{o}(X) \rightarrow \mathfrak{o}(Z)$ .

**Lemma 4.** *If  $i$  is the natural embedding of the space  $X$  into  $Z$ , then  $i^*$  and  $P$  are mutually inverse isomorphisms of the algebras  $\mathfrak{o}(X)$  and  $\mathfrak{o}(Z)$ .*

Denote by  $\mathfrak{M}(X)$  the set of all homomorphisms  $\omega$  of the algebra  $\mathfrak{o}(X)$  into  $C$  satisfying the condition  $\omega(1) = 1$ . To each point  $z \in Z$  assign the mapping  $\omega_z : \mathfrak{o}(X) \rightarrow C$ , defined by the formula

$$\omega_z(f) = (Pf)(z) \quad (f \in \mathfrak{o}(X)).$$

It follows from Lemma 4 that  $\omega_z \in \mathfrak{M}(X)$ . Our main result consists in the following.

**Theorem 1.** *The mapping  $\Omega : Z \rightarrow \mathfrak{M}(X)$ , defined by the formula  $\Omega(z) = \omega_z$ , is a one-to-one mapping onto.*

Let us outline the proof of this theorem. First of all, in the special case  $\Omega = \{e\}$ , Theorem 1 is easily derived from the results of Chapter VI of the book <sup>(3)</sup>. In the general case, denote by  $p$  the natural projection of the group  $\mathfrak{G}$  onto  $Z$ . It is easy to see that  $p^*(\mathfrak{o}(X)) \subset \mathfrak{o}(\mathfrak{K})$ . Therefore there arises a mapping ...

$$\bar{p} : \mathfrak{M}(\mathfrak{K}) \rightarrow \mathfrak{M}(X),$$

which assigns to each  $\omega \in \mathfrak{M}(\mathfrak{K})$  the homomorphism  $\omega p^*$ . It turns out that the diagram

$$\begin{array}{ccc} \mathfrak{G} & \xrightarrow{\tilde{\Omega}} & \mathfrak{M}(\mathfrak{K}) \\ p \downarrow & & \downarrow \bar{p} \\ Z & \xrightarrow{\Omega} & \mathfrak{M}(X) \end{array}$$

is commutative (by  $\tilde{\Omega}$  is denoted the mapping analogous to the mapping  $\Omega$ ). Since  $\tilde{\Omega}$  is onto, in order to prove that  $\Omega$  is onto it is enough to show that  $\bar{p}$  is onto. The latter follows from the following lemma:

**Lemma 5.** Let  $A$  and  $B$  be commutative algebras with identities and with a finite number of generators over the field  $C$ ; let  $\alpha : A \rightarrow B$  be such a homomorphism that  $\alpha(1) = 1$ . Suppose that there exists a linear mapping  $\beta : B \rightarrow A$  such that  $\beta(1) = 1$  and  $\beta(\alpha(x) \cdot y) = x \cdot \beta(y)$  ( $x \in A$ ,  $y \in B$ ). Then every homomorphism  $\omega : A \rightarrow C$  can be represented in the form  $\omega = \tilde{\omega}\alpha$ , where  $\tilde{\omega}$  is a homomorphism of the algebra  $B$  into  $C$ .

Lemma 5 must be applied to the case  $A = \mathfrak{o}(X)$ ,  $B = \mathfrak{o}(\mathfrak{K})$ ,  $\alpha = p^*$ . We define the mapping  $\beta$  by the formula

$$(\beta f)(ko) = \int_{\mathfrak{L}} f(kl) dl \quad (f \in \mathfrak{o}(\mathfrak{K})),$$

where integration is with respect to the invariant measure on  $\mathfrak{L}$ , normalized by the condition

$$\int_{\mathfrak{L}} dl = 1.$$

The one-to-one character of the mapping  $\Omega$  follows from Lemma 3.

**3.** Let  $\mathfrak{G}$  be a group, and let  $\mathfrak{G}'$  and  $\mathfrak{G}''$  be its subgroups. We say that the triple  $(\mathfrak{G}, \mathfrak{G}', \mathfrak{G}'')$  is a **decomposition** if every element  $g \in \mathfrak{G}$  can be represented in the form  $g = g' \cdot g''$ , where  $g' \in \mathfrak{G}'$ ,  $g'' \in \mathfrak{G}''$ . The study of decompositions of Lie groups is equivalent to the study of inclusions between transitive Lie groups of transformations <sup>(4)</sup>. Let  $\mathfrak{K}', \mathfrak{K}''$  be connected closed subgroups of a connected compact Lie group  $\mathfrak{K}$ ; let  $\mathfrak{G}', \mathfrak{G}'', \mathfrak{G}$  be the complex envelopes of these groups. Then  $\mathfrak{G}'$  and  $\mathfrak{G}''$  are identified with subgroups of the group  $\mathfrak{G}$ . The triple  $(\mathfrak{G}, \mathfrak{G}', \mathfrak{G}'')$  will be called the **complex envelope** of the triple  $(\mathfrak{K}, \mathfrak{K}', \mathfrak{K}'')$ .

**Theorem 2.** *The complex envelope of a decomposition is a decomposition.*

Let a decomposition  $(\mathfrak{G}, \mathfrak{G}', \mathfrak{G}'')$  be given, where  $\mathfrak{G}$  is a semisimple complex Lie group, and  $\mathfrak{G}', \mathfrak{G}''$  are its connected closed complex subgroups. Let  $\mathfrak{G}'_0, \mathfrak{G}''_0$  be maximal semisimple subgroups of the groups  $\mathfrak{G}', \mathfrak{G}''$ . Then the triple  $(\mathfrak{G}, \mathfrak{G}'_0, \mathfrak{G}''_0)$  is also a decomposition. The groups  $\mathfrak{G}', \mathfrak{G}''$  may be replaced by conjugate subgroups in  $\mathfrak{G}$  in such a way that the decomposition  $(\mathfrak{G}, \mathfrak{G}'_0, \mathfrak{G}''_0)$  is the complex envelope of some decomposition  $(\mathfrak{K}, \mathfrak{K}', \mathfrak{K}'')$ , where  $\mathfrak{K}, \mathfrak{K}', \mathfrak{K}''$  are semisimple compact Lie groups.

**4.** From Theorem 1 the following is derived.

**Theorem 3.** *Let  $Z$  be the complex envelope of the compact homogeneous space  $X$ , and let  $\Phi : X \rightarrow V$  be a one-to-one holomorphic covariant mapping. Then the corresponding covariant mapping  $\Psi$  maps  $Z$  one-to-one and biholomorphically onto a nonsingular algebraic subvariety of the space  $V$ . If one introduces on  $Z$ , by means of the mapping  $\Psi$ , the structure of an algebraic variety, then the algebra of regular functions on  $Z$  will coincide with  $\mathfrak{o}(Z)$ , and every complex-analytic covariant mapping of the space  $Z$  into a finite-dimensional vector space will be a rational mapping.*

We say that a complex Lie group is **admissible** if it is the complex envelope of some compact group. Let  $\mathfrak{H}$  be an admissible complex-analytic subgroup of an admissible group  $\mathfrak{G}$ . Then  $\mathfrak{H}$  is closed. It follows from Theorem 3 that the manifold  $\mathfrak{G}/\mathfrak{H}$  is a Stein manifold <sup>(5)</sup>. It turns out that this result admits the following converse:

**Theorem 4.** *Let  $\mathfrak{G}$  be an admissible complex Lie group, and let  $\mathfrak{H}$  be its closed complex-analytic subgroup. If  $\mathfrak{G}/\mathfrak{H}$  is a Stein manifold, then the connected component of the identity in the group  $\mathfrak{H}$  is admissible.*

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*Note: Figure translations are in progress. See original paper for figures.*

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