MATHEMATICS

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

V. F. ZHDANOVICH

FORMULAS FOR THE ZEROS OF DIRICHLET POLYNOMIALS AND QUASIPOLYNOMIALS

(Presented by Academician I. G. Petrovskii, 28 VI 1960)

1. From the work of M. G. Krein and B. Ya. Levin ([1]) (see also ([2])) it follows that for the zeros of the Dirichlet polynomial

$$e^{\alpha_0\lambda} + a_1 e^{\alpha_1\lambda} + \dots + a_{p-1} e^{\alpha_{p-1}\lambda} + a_p, \tag{1}$$

where α_s $(s = 0, 1, \dots, p-1)$ are real numbers; $\alpha_{s-1} > \alpha_s > 0$ $(s = 1, 2, \dots, p)$, a_s $(s = 1, 2, \dots, p)$ are complex numbers, $a_p \neq 0$, the formula

$$\lambda_n = \frac{2\pi n i}{\alpha_0} + \psi(n), \tag{2}$$

$(n = 0, \pm 1, \dots)$, holds, where $\psi(n)$ is a certain bounded complex-valued function of n, which is not computed in the cited works. For the computation of the function $\psi(n)$ in the case of the polynomial (1), the following scheme may be proposed. Let $P_1(t_1^{(1)}, t_2^{(1)}, \dots, t_{p-1}^{(1)})$ be a point of the $(p-1)$-dimensional torus T_{p-1} $(0 \le t_s < 2\pi,\ s = 1, 2, \dots, p-1)$. In a certain neighborhood of the point $R_1(e^{it_1^{(1)}}, e^{it_2^{(1)}}, \dots, e^{it_{p-1}{}^{(1)}})$ of the $(p-1)$-dimensional complex space, find the solution $z = z(\mu_1, \mu_2, \dots, \mu_{p-1})$ of the equation

$$e^{\alpha_0 z} + \sum_{s=1}^{p-1} a_s \mu_s e^{\alpha_s z} + a_p = 0, \tag{3}$$

normalized by the condition

$$0 \le \operatorname{Im} z(e^{it_1^{(1)}}, e^{it_2^{(1)}}, \dots, e^{it_{p-1}{}^{(1)}}) < \frac{2\pi}{\alpha_0}. \tag{4}$$

We shall assume that the point P_1 was chosen so that the point R_1 is not a branch point of the solution of equation (3), and we compute the coefficients in the expansion of the function $z = z(\mu_1, \mu_2, \dots, \mu_{p-1})$ into a power series at this point by the known formulas for the coefficients of the Taylor series of an implicit function. We analytically continue the function thus obtained along the torus T_{p-1} to a set $E_1 \subset T_{p-1}$ in such a way that the resulting analytic continuation $z = z_1(\mu_1, \mu_2, \dots, \mu_{p-1})$ satisfies on E_1 the inequality

$$0 \leq \operatorname{Im} z_1(\mu_1, \mu_2, \dots, \mu_{p-1}) < \frac{2\pi}{\alpha_0} \tag{5}$$

and E_1 admits no enlargement preserving the properties indicated above; we also adjoin to the set E_1 those branch points which lie on the boundary of the set E_1 and to which the function $z = z_1(\mu_1, \mu_2, \dots, \mu_{p-1})$ is continued continuously with preservation of condition (5). On the complement of E_1 in T_{p-1} choose a point $P_2(t_1^{(2)}, t_2^{(2)}, \dots, t_{p-1}^{(2)})$ and construct in an analogous way the set E_2 and the function $z = z_2(\mu_1, \mu_2, \dots, \mu_{p-1})$, etc. It is not difficult to prove that after a finite number of steps the whole torus will be covered by the sets E_s $(s = 1, 2, \dots, k)$, on which

continuous functions $z = z_s(\mu_1, \mu_2, \dots, \mu_{p-1})$, analytic at all interior points of the set E_s, are defined. These functions give a function defined on the whole torus $z = \psi(e^{it_1}, e^{it_2}, \dots, e^{it_{p-1}})$, satisfying the condition $0 \leq \operatorname{Im} \psi < \frac{2\pi}{a_0}$. Then for the zeros of the Dirichlet polynomial (1) the formula

$$\lambda_n = \frac{2\pi n i}{a_0} + \psi\left(\exp\left[i\frac{2\pi\alpha_1 n}{a_0}\right], \exp\left[i\frac{2\pi\alpha_2 n}{a_0}\right], \dots, \exp\left[i\frac{2\pi\alpha_{p-1} n}{a_0}\right]\right) \tag{6}$$

$$(n = 0,\ \pm 1, \dots).$$

2. We shall now obtain formulas for sufficiently large, in modulus, zeros of the quasipolynomial

$$\sum_{s=0}^{p} a_s \lambda^{k_s} e^{\alpha_s \lambda}, \tag{7}$$

where k_s $(s = 0, 1, \dots, p)$ are nonnegative integers, with $\min k_s = 0$ $(s = 0, 1, \dots, p)$; α_s and a_s $(s = 0, 1, \dots, p)$ are the same as for the polynomial (1); $\alpha_p = 0$, $a_0 = 1$, $a_p \neq 0$. The distribution of zeros of quasipolynomials was studied by N. G. Chebotarev and N. N. Meiman ([3]), L. S. Pontryagin ([4]), and others.

We shall seek the zeros of the quasipolynomial (7) in the form

$$\lambda = \gamma \ln|\tau| + i\tau + \lambda_0, \tag{8}$$

where γ and τ are real, and λ_0 is complex. Substituting into the quasipolynomial (7) and setting it equal to zero, we obtain

$$\sum_{s=0}^{p} a_s \left(i \operatorname{sgn} \tau + \frac{\lambda_0}{|\tau|} + \frac{\gamma \ln |\tau|}{|\tau|} \right)^{k_s} |\tau|^{\alpha_s \gamma + k_s} e^{i\alpha_s \tau} e^{\alpha_s \lambda_0} = 0. \tag{9}$$

For fixed γ and λ_0 and $|\tau| \to \infty$, the left-hand side of equality (9) can tend to zero only for those γ for which the numbers $\delta_s = \alpha_s \gamma + k_s$ $(s = 0, 1, \dots, p)$ are not all distinct. Let, for the broken line $y = \max_s(\alpha_s x + k_s)$ $(s = 0, 1, \dots, p)$, the vertices be the points $P_r(\gamma_r, \delta_r)$ $(r = 1, 2, \dots, m \leq p)$. At each such vertex the numbers δ_s $(s = 0, 1, \dots, p)$ will not all be distinct, and each such vertex will correspond to an asymptotic formula for the zeros of the quasipolynomial. If two straight lines intersect at one vertex, for example $y = \alpha_{s_1} x + k_{s_1}$ and $y = \alpha_{s_2} x + k_{s_2}$ $(s_1 < s_2)$, then these formulas are obtained as follows. If equation (9) is divided by $|\tau|^{\delta_{s_1}}$, then, since $\alpha_s \gamma + k_s < \delta_{s_1}$ $(s = 0, 1, \dots, p,\ s \neq s_1,\ s \neq s_2)$, as $|\tau| \to \infty$ we obtain, for determining τ and λ_0, an equation of the form

$$a_{s_1} (i \operatorname{sgn} \tau)^{k_{s_1}} \exp[\alpha_{s_1}(\lambda_0 + i\tau)] + a_{s_2} (i \operatorname{sgn} \tau)^{k_{s_2}} \exp[\alpha_{s_2}(\lambda_0 + i\tau)] + O\left(\frac{\ln |\tau|}{|\tau|} \right) = 0, \tag{10}$$

from which, for $\tau > 0$, we obtain

$$\lambda_0 + i\tau = \frac{2\pi n i}{\alpha_{s_1} - \alpha_{s_2}} + \frac{i}{\alpha_{s_1} - \alpha_{s_2}} \ln \left(-\frac{a_{s_2}}{a_{s_1}} i^{k_{s_2} - k_{s_1}} \right) + \varepsilon_n \tag{11}$$

$$(n = 1, 2, \dots), \quad \text{where} \quad \varepsilon_n = O\left(\frac{\ln n}{n} \right).$$

Put

$$\tau = \tau_n = \frac{2\pi n}{\alpha_{s_1} - \alpha_{s_2}}; \qquad \lambda_0 = \frac{i}{\alpha_{s_1} - \alpha_{s_2}} \ln \left(-\frac{a_{s_2}}{a_{s_1}} i^{k_{s_2} - k_{s_1}} \right) + \varepsilon_n$$

and substitute into equation (9). If by $\beta_1, \beta_2, \dots, \beta_q$ we denote a basis of the system of numbers

$$1,\ \frac{\alpha_s - \alpha_{s_1}}{\alpha_{s_2} - \alpha_{s_1}} (k_{s_1} - k_{s_2}) + k_s - k_{s_1} \tag{12}$$

$$(s = 0, 1, \dots, p,\ s \neq s_1,\ s \neq s_2)$$

(i.e., a system of numbers such that each of the numbers in the system (12) is a linear combination of these numbers with nonnegative integer coefficients), and if instead of $\ln n/n$ we write μ_0, and instead of $1/n^{\beta_s}$ we write μ_s $(s = 1, 2, \dots, q)$, then for determining ε_n we obtain from equation (6) an equation of the form

$$F\left(\varepsilon_n,\ \mu_0, \mu_1, \dots, \mu_q; \exp\left[\frac{2\pi n(\alpha_0 - \alpha_{s_2})i}{\alpha_{s_1} - \alpha_{s_2}}\right], \dots, \exp\left[\frac{2\pi n(\alpha_p - \alpha_{s_2})i}{\alpha_{s_1} - \alpha_{s_2}}\right]\right) = 0, \tag{13}$$

which turns into equation (9) when $\mu_0 = \ln n/n,\ \mu_s = 1/n^{\beta_s}$ $(s = 1, 2, \dots, q)$. From this equation we determine $\varepsilon_n = \varepsilon_n(\mu_0, \mu_1, \dots, \mu_q)$ in the form of a power series

$$\varepsilon_n = \sum_{k_0+k_1+\dots+k_q=1}^{\infty} a_{k_0 k_1 \dots k_q}(n) \mu_0^{k_0} \mu_1^{k_1} \cdots \mu_q^{k_q}, \tag{14}$$

where the Taylor coefficients $a_{k_0 k_1 \dots k_q}(n)$ are analytic functions of

$$\eta_s = \exp\left[\frac{2\pi n(\alpha_s - \alpha_{s_2})}{\alpha_{s_1} - \alpha_{s_2}}\, i\right] \quad (s = 0, 1, \dots, p,\ s \neq s_1,\ s \neq s_2).$$

This series converges for sufficiently small $|\mu_s|$ $(s = 0, 1, \dots, q)$, uniformly in n. Substituting in (14) $\mu_0 = \ln n/n,\ \mu_s = 1/n^{\beta_s}$ $(s = 1, 2, \dots, q)$, we obtain ε_n. Formulas (14), (11), and (8) give the formula

$$\lambda_n = \gamma \ln \tau_n + i\tau_n + \lambda_0 + \sum_{k_0+k_1+\dots+k_q=1}^{\infty} a_{k_0 k_1 \dots k_q}(n) \left(\frac{\ln n}{n}\right)^{k_0} \left(\frac{1}{n^{\beta_1}}\right)^{k_1} \cdots \left(\frac{1}{n^{\beta_q}}\right)^{k_q}, \tag{15}$$

where

$$\gamma = \frac{k_{s_2} - k_{s_1}}{\alpha_{s_1} - \alpha_{s_2}}, \quad \tau_n = \frac{2\pi n}{\alpha_{s_1} - \alpha_{s_2}}, \quad \lambda_0 = \frac{i}{\alpha_{s_1} - \alpha_{s_2}} \ln\left(\frac{a_{s_2}}{a_{s_1}}\, i^{k_{s_2} - k_{s_1}}\right),$$

and where the series converges absolutely and uniformly for $n > n_0$, with n_0 sufficiently large.

A completely analogous formula is obtained for $\tau < 0$, and also for other vertices at which two straight lines intersect. If, however, more than two straight lines intersect at one vertex, for example $y = \alpha_{s_j} x + k_{s_j}$ $(j = 1, 2, \dots, r,\ s_j < s_{j+1})$, then instead of equation (10) we obtain the equation

$$\sum_{j=1}^{r} a_{s_j}(i \operatorname{sgn}\tau)^{k_{s_j}} \exp[\alpha_{s_j}(\lambda_0 + i\tau)] + O\left(\frac{\ln|\tau|}{|\tau|}\right) = 0,$$

and for γ, τ_n, and λ_0 we obtain the expressions

$$\gamma = \frac{k_{s_r} - k_{s_1}}{\alpha_{s_1} - \alpha_{s_r}}, \quad \tau_n = \frac{2\pi n}{\alpha_{s_1} - \alpha_{s_r}}, \quad \lambda_0 = \psi(n),$$

where the function $\psi(n)$ is taken from formula (2). Instead of formula (15), for $n > 0$ the formula will be

$$\lambda_n = \gamma \ln \tau_n + i\tau_n + \psi(n) + \sum_{k_0+k_1+\cdots+k_q=1}^{\infty} a_{k_0 k_1 \dots k_q}(n) \left(\frac{\ln n}{n}\right)^{k_0} \left(\frac{1}{n^{\beta_1}}\right)^{k_1} \cdots \left(\frac{1}{n^{\beta_q}}\right)^{k_q}. \tag{16}$$

$n > n_0$, where the Taylor coefficients $a_{k_0 k_1 \dots k_q}(n)$ are analytic functions of $p-1$ variables

$$\eta_s = \exp\left[\frac{2\pi n(\alpha_s - \alpha_{s_r})}{\alpha_{s_1} - \alpha_{s_r}} i\right] \quad (s = 0, 1, \dots, p, \; s \neq s_1, \; s \neq s_2)$$

and of the variable $\eta = \psi(n)$. An analogous formula also holds for $n < 0$.

Thus, we obtain:

Theorem 1. *To each vertex of the broken line $y = \max_s(\alpha_s x + k_s)$ $(s = 0, 1, \dots, p)$ there corresponds a sequence of zeros of the quasipolynomial (7). If at the vertex two of the lines $y = \alpha_s x + k_s$ $(s = 0, 1, \dots, p)$ intersect, then for zeros of the quasipolynomial (7) sufficiently large in modulus formula (15) holds; but if more than two lines intersect at the vertex, then for zeros sufficiently large in modulus formula (16) holds.*

3. The method set forth for determining zeros is also applicable in the case when, in the polynomials (1) and (7), the exponents α_s $(s = 0, 1, \dots, p)$ are complex.

Theorem 2. *If the polynomial (1) has complex exponents α_s $(s = 0, 1, \dots, p)$, then to each side of the polygon that is the convex hull of the numbers α_s $(s = 0, 1, \dots, p)$ there corresponds a sequence of zeros λ_n $(n = 1, 2, \dots)$, and for these zeros the formula*

$$\lambda_n = e^{-i\theta} \frac{2\pi n i}{h} + \varphi(n) \tag{17}$$

holds $(n = 1, 2, \ldots)$, *where* θ *is the angle between the side of the polygon and the real axis;* h *is the length of this side;* $\varphi(n)$ *is a bounded complex-valued function admitting the representation*

$$\varphi(n) = \sum_{k=0}^{\infty} b_k(n) \left(\frac{1}{e^{\alpha n}}\right)^k,$$

where $\alpha > 0$, *and* $b_k(n)$ $(k = 0, 1, \ldots)$ *are the Taylor coefficients of a certain analytic function with parameter* n.

Theorem 3. *If the quasipolynomial (7) has complex exponents* α_s $(s = 0, 1, \ldots, p)$, *then to each side of the polygon that is the convex hull of the numbers* α_s $(s = 0, 1, \ldots, p)$ *there corresponds a finite number* j_0 *of sequences* $\lambda_n^{(j)}$ $(j = 1, 2, \ldots, j_0;\ n = 1, 2, \ldots)$ *of zeros of the quasipolynomial (7), the number* j_0 *being equal to the number of vertices of a certain broken line (for each side, its own). For the zeros* $\lambda_n^{(j)}$ *of each sequence the formula*

$$\lambda_n^{(j)} = e^{-i\theta}(\gamma_j \ln \tau_n^{(j)} + i\tau_n^{(j)}) + \varphi_j(n) \tag{18}$$

holds $(j = 1, 2, \ldots, j_0;\ n = 1, 2, \ldots)$, *where* θ *is the angle between the side of the polygon and the real axis;* $\gamma_j, \tau_n^{(j)}$ $(j = 1, 2, \ldots, j_0;\ n = 1, 2, \ldots)$ *are real numbers;* $\tau_n^{(j)} \to \infty$ *as* $n \to \infty$; $\varphi_j(n)$ $(j = 1, 2, \ldots, j_0)$ *are bounded complex-valued functions admitting a representation of the form*

$$\varphi_j(n) = \sum_{p_1+p_2+k_1+\cdots+k_q=1}^{\infty} a_{p_1 p_2 k_1 \ldots k_q}(n) \left(\frac{1}{e^{\alpha n}}\right)^{p_1} \left(\frac{\ln n}{n_i}\right)^{p_2} \left(\frac{1}{n^{\beta_1}}\right)^{k_1} \cdots \left(\frac{1}{n^{\beta_q}}\right)^{k_q}$$

$$(j = 1, 2, \ldots, j_0), \quad \alpha > 0, \quad \beta_s > 0\ (s = 1, 2, \ldots, q).$$

Received
27 VI 1960

REFERENCES

1. M. G. Krein, B. Ya. Levin, *Dokl. Akad. Nauk SSSR*, **64**, 3 (1949).

2. B. Ya. Levin, *Distribution of the Roots of Entire Functions*, Moscow, 1956.

3. N. G. Chebotarev, N. N. Meiman, *Trudy Mat. Inst. im. V. A. Steklova, Akad. Nauk SSSR*, **26**, 24 (1949).

4. L. S. Pontryagin, *Izv. Akad. Nauk SSSR, Ser. Mat.*, **6**, 115 (1942).