MATHEMATICS

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

L. A. SHOR

ON UNIQUE DETERMINACY AND ON THE FLEXIBILITY OF CERTAIN CONVEX SURFACES WITH BOUNDARY

(Presented by Academician P. S. Aleksandrov on 15 IV 1960)

We shall say that a convex surface F, homeomorphic to an open disk, is a **surface of type** K if its completion to the boundary $\bar{F}$ of its convex hull is a convex domain in the plane. The aim of the present paper is to establish effectively verifiable necessary and sufficient conditions both for unique determinacy and for flexibility of surfaces of type K in the class of all convex surfaces. The corresponding conditions for convex polyhedra were established by A. D. Aleksandrov ([1]), pp. 233-236.

1. Let us recall that a convex surface F is called **uniquely determined** in the class of all convex surfaces (hereafter simply uniquely determined) if every convex surface F' isometric to the surface F is equal to it or is its mirror image. A convex surface F is called **flexible** in the class of all convex surfaces (hereafter simply flexible) if it can be included in a continuous family of isometric, but not equal to it, convex surfaces F_t $(t \in [0,1],\ F_0 = F)$.

As A. V. Pogorelov showed ([2]), every general closed convex surface is uniquely determined. For general convex surfaces with boundary, only the necessary condition for unique determinacy established by A. S. Leibin ([3]) is known. Using A. S. Leibin' s condition, one can obtain the following theorem, which will be needed below.

Theorem 1. *If a convex surface F contains a uniquely determined convex surface Φ, then the surface F itself is uniquely determined.*

2. Let F be a convex surface homeomorphic to an open disk, bounded by a simple closed curve Γ having finite variation of rotation. Let Q be a domain, homeomorphic to a closed disk and, generally speaking, multiply connected, in the plane, bounded by a curve L having finite variation of rotation. Suppose that the curve L admits a mapping ψ onto the curve Γ satisfying the conditions of the gluing theorem ([4]), pp. 289-293. The manifold $F+Q$, glued from the surface F and the domain Q, is realized by some closed convex surface $\tilde{F}$. The construction of the surface $\tilde{F}$ from the surface F, the domain Q, and the mapping ψ is called the **gluing of** Q **to** F and is denoted by (Q, ψ). The gluing is called **trivial** if

Fig. 1

Figure 1: Fig. 1

the surface $\tilde{F}$ coincides with the boundary $\bar{F}$ of the convex hull of the surface F. Two gluings (Q, ψ) and (Q', ψ') to the surface F are considered identical if and only if the manifolds $F + Q$ and $F + Q'$ are isometric and, moreover, the isometric mapping f of the manifold $F + Q$ onto the manifold $F + Q'$ maps F onto itself and Q onto Q'.

Theorem 2. *In order that a convex surface F, homeomorphic to an open disk, be uniquely determined, it is necessary and sufficient that it admit no nontrivial gluing.*

Theorem 2 is a generalization of a well-known result of A. D. Aleksandrov for polyhedra (^{1}), pp. 230-231). It reduces the question of the unique determination of a convex surface to the question of the unique determination of a plane, generally speaking, many-sheeted domain satisfying the prescribed gluing conditions. From Theorem 2 there follows directly the unique determination of convex surfaces homeomorphic to an open disk and with total curvature equal to 4π.

Fig. 1

3. It is said that there is a **deformation of a trivial gluing** if to each value of the parameter $t \in [0, 1]$ there is assigned a gluing (Q_t, ψ_t) of the domain Q_t to the surface F, given by a mapping ψ_t of the curve L_t onto the curve Γ, and the following conditions are satisfied:

1) for every $t_0 \in [0, 1]$ and $t \to t_0$, the domains Q_t converge to the domain Q_{t_0} in such a way that, for every point $x \in \Gamma$,

$$\lim_{t \to t_0} \psi_t^{-1}(x) = \psi_{t_0}^{-1}(x);$$

2) not all the gluings (Q_t, ψ_t) are the same;
3) the gluing (Q_0, ψ_0) is trivial.

The following theorem holds, analogous to the well-known theorem of A. D. Aleksandrov for polyhedra (^{1}), pp. 232-233).

Theorem 3. *In order that a convex surface F, homeomorphic to an open disk, be bendable, it is necessary and sufficient that there exist a deformation of its corresponding trivial gluing.*

4. We shall agree to denote corresponding points of the curves L and Γ under a gluing by the same letters. The conical points of the surface $\bar{F}$ lying on the curve Γ will be denoted by A_i $(i = 1, 2, ...)$, and we shall call them points of type A. Denote the rotation of the curve Γ on the surface $\bar{F}$ at

the point A_i, from the side of the surface F, by φ_{i1}, and the rotation of the curve L at the point A_i, from the side of the domain Q, by φ_{i0}. The area of the spherical image of a set M on the surface $\bar{F}$ will be denoted by $\omega(M)$.

Theorem 4. *In order that a convex surface F of type K be uniquely determined, it is necessary and sufficient that the following two conditions be satisfied:*

4.1. *The area of the spherical image of the curve Γ on the surface $\bar{F}$ is concentrated in no more than one point, i.e. either $\omega(\Gamma) = 0$, or $\omega(\Gamma) = \omega(A_1) > 0$.*

4.2. *If $\omega(\Gamma) > 0$, then the rotation, from the side of the domain Q, of each open segment of the curve L containing the point A_1 is greater than φ_{11}.*

Moreover, if conditions 4.1 and 4.2 are not satisfied, then the surface F is bendable.

5. In the proof of sufficiency, the following lemma is used essentially:

Lemma. *Let CD and CE be plane curves of bounded variation of rotation, having a common point C and lengths equal to σ (Fig. 1). Suppose that the left rotations $\varphi(s)$ and $\theta(s)$ of the open segments of these curves, corresponding to the variable arc s $(0 < s \leq \sigma)$, measured on the curve CD from the point D and on the curve CE from the point E, are nonnegative and monotonically nondecreasing. The angle* formed by the half-tangent to the curve CE at the point C with the half-tangent to the curve CD at the same point will be denoted by γ_0. Denote by n the normal* to the curve CD at the point D, directed toward the convexity of this curve. Let the rotations $\varphi(s)$ and $\theta(s)$ satisfy the conditions:

5.1. The function $\gamma(s)$, defined by the equalities $\gamma(0) = 0$, $\gamma(s) = \theta(s) - \varphi(s)$ for $0 < s < \sigma$, $\gamma(\sigma) = \theta(\sigma) - \varphi(\sigma) + \gamma_0$, is monotonically nondecreasing.

5.2. $0 < \gamma(\sigma) < 2(\pi - \varphi(\sigma))$.

Then:

1) the angle β, formed by the normal n with the vector DE, satisfies the inequality

$$\pi - \frac{1}{2}\gamma(\sigma) \leqslant \beta < \pi + \varphi(\sigma);$$

2) the angle $\beta = \pi - \frac{1}{2}\gamma(\sigma)$ if and only if the function $\gamma(s)$ has the form

---

* Here and in what follows, by the angle formed by one ray with another is meant the angle through which the first ray must be rotated counterclockwise in order for it to coincide with the second.

$$\gamma(s) = \begin{cases} 0, & 0 < s \leqslant s', \\ \gamma(\sigma), & s' < s \leqslant \sigma, \end{cases}$$

where $s' \in (0, \sigma]$ is such that $\varphi(s') = 0$.

6. Let a convex surface F of type K satisfy conditions 4.1 and 4.2. Construct a certain convex domain G, containing the domain Q and satisfying the conditions:

6.1. The boundaries of the domains G and Q have a common segment, an interior point of which is the point A_1 of type A.

6.2. On the boundary R of the domain G there is a point B such that the rotations of the open segments R_1 and R_2, into which the points A_1 and B divide the curve R, are equal to ${}^3/_4(\pi - \varphi_{10})$.

6.3. Let Φ be some convex surface homeomorphic to an open disk, which the domain G completes to the boundary of its convex shell $\overline{\Phi}$. Suppose moreover that the area of the spherical image of the boundary of the surface Φ on the surface $\overline{\Phi}$ is equal to $\omega(A_1)$ and is concentrated at the point corresponding, under gluing, to the point A_1 of the curve R.

If the domain G is uniquely determined by the conditions of gluing to the surface Φ, then, by Theorem 2, the surface Φ is uniquely determined. But then, according to Theorem 1, the surface $\Phi_1 = \Phi + (G - Q)$ is also uniquely determined, and, by Theorem 2, the domain $Q \subset G$ is uniquely determined by the conditions of gluing to the surface Φ_1. Since, by 6.1 and 6.3, the conditions of gluing of the domain Q to the surfaces F and Φ_1 coincide, it follows that the surface F must be uniquely determined.

7.1. Suppose, contrary to what is being proved, that along with the domain G, the conditions of gluing to the surface Φ are satisfied by some incongruent domain G'. The elements of the domain G' corresponding under gluing to the elements of the domain G will be marked with a prime.

7.2. It is not difficult to prove that for the boundary R' of the domain G' the following alternative holds: either the curve R' has no multiple points, or the point A_1' is combined with some other point of the curve R'.

7.3. Arrange the domains G and G' so that the points B and B' of these domains and the half-tangents to the curves R_1 and R_1' at these points coincide. Suppose, for definiteness, that in moving from the point A_1 along the curve R_1, the interior points of the domain G lie on the left. Denote by γ_j the difference of the rotations of the curves R_j' and R_j $(j = 1, 2)$; denote by δ the angle formed by the half-tangent to the curve R_2 at the point B with the half-tangent to the curve R_2' at the point B'. Obviously,

$$\gamma_1 + \gamma_2 + \delta \leqslant \omega(A_1) = \varphi_{10} + \varphi_{11} < 2\pi - 2 \cdot {}^3/_4(\pi - \varphi_{10}). \tag{1}$$

Hence, and from 6.2 and 7.1, it follows that the curves R_1 and R_1', R_2 and R_2' satisfy the conditions of the lemma.

8. Denote by β the angle formed by the exterior normal to the curve R_1 at the point A_1 with the vector A_1A_1'. From the consideration of the curves R_1 and R_1', by virtue of §§ 7.2, 7.3 and the lemma, there follows the inequality

$$\max\left(\frac{\pi}{2},\, \pi - \frac{1}{2}\gamma_1\right) \leqslant \beta < \pi + \frac{3}{4}(\pi - \varphi_{10}). \tag{2}$$

Similarly, from the consideration of the curves R_2 and R_1', we obtain the inequality

$$\frac{1}{4}(\pi - \varphi_{10}) < \beta \leqslant 2\pi - \varphi_{10} - \max\left[\frac{\pi}{2},\, \pi - \frac{1}{2}(\gamma_2 + \delta)\right]. \tag{3}$$

It follows from 7.2 that the difference between the lower estimate of β in inequality (2) and the upper estimate of β in inequality (3) is equal to $\dfrac{\varphi_{10} - \varphi_{11}}{2}$. Since $\varphi_{10} \geqslant \varphi_{11}$, and when $\varphi_{10} = \varphi_{11}$, by virtue of condition 4.2 of Theorem 4 and assertion 2 of the lemma, at least one of these estimates is strict, it follows that inequalities (2) and (3) are incompatible.

The sufficiency of the condition of Theorem 4 has been proved. The proof of necessity is carried out on the basis of Theorem 3 by an effective construction of a continuous deformation of the domain Q preserving the conditions of gluing.

Voroshilovgrad Mining and Metallurgical
Institute

Received
15 IV 1960

REFERENCES

[1] A. D. Aleksandrov, *Convex polyhedra*, Moscow-Leningrad, 1950.
[2] A. V. Pogorelov, *Unique determination of general convex surfaces*, Kiev, 1952.
[3] A. S. Leibin, UMN, 5, issue 5 (39) (1950).
[4] A. D. Aleksandrov, *Intrinsic geometry of convex surfaces*, Moscow-Leningrad, 1948.

Note: Figure translations are in progress. See original paper for figures.

Source: Math-Net.Ru and CyberLeninka. Machine translation. Verify with the original.