MATHEMATICS

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Abstract

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ON THE APPLICATION OF NUMBER-THEORETIC METHODS OF INTEGRATION IN THE CASE OF NONPERIODIC FUNCTIONS

(Presented by Academician S. L. Sobolev, 3 XII 1959)

1. In numerous works devoted to the approximate computation of multiple integrals that have appeared recently, quadrature formulas have been constructed, in particular, for functions $f(x_1, \dots, x_s)$ belonging to the class $E_s(\alpha)$, defined by the conditions:

$$f(x_1, \dots, x_s) = \sum_{m_1, \dots, m_s = -\infty}^{\infty} c(m_1, \dots, m_s) e^{2\pi i (m_1 x_1 + \cdots + m_{sx} s)},$$

$$c(m_1, \dots, m_s) = O\left((\overline{m}_1 \cdots \overline{m}_s)^{-\alpha}\right), \qquad \overline{m}_i = \max(1, |m_i|), \qquad \alpha > 1.$$

Let us denote by $H_s(\alpha, c)$ the class of functions defined by the conditions

$$\left| \frac{\partial^n f(x_1, \dots, x_s)}{\partial x_1^{\gamma_1} \cdots \partial x_s^{\gamma_s}} \right| < c, \quad 0 \le n \le \alpha s, \qquad 0 \le \gamma_i \le \alpha, \qquad \gamma_1 + \cdots + \gamma_s = n.$$

Periodic functions from $H_s(\alpha, c)$ also belong to $E_s(\alpha)$. However, it is possible to construct quadrature formulas acting on the whole class $H_s(\alpha, c)$.

Theorem 1. *Let $f(x_1, \dots, x_s)$ belong to $H_s(\alpha, c)$, $\alpha > 1$; let $\tau_\alpha(z)$ be a certain function satisfying the conditions:*

1)
$$0 = \tau_\alpha(0) < \tau_\alpha(z') < \tau_\alpha(z'') < \tau_\alpha(1) = 1, \quad 0 < z' < z'' < 1;$$

2)
$$\left|\tau_\alpha^{(k)}(z)\right| < A, \quad 0 \le k \le \alpha + 1;$$

3)
$$\tau_\alpha^{(k)}(0) = \tau_\alpha^{(k)}(1) = 0, \quad 1 \le k \le \alpha.$$

Then there exist integers $a_1, \dots, a_s$, $a_i = a_i(N)$, such that

$$R = \left| \frac{1}{N} \sum_{k=1}^{N} f\left[\tau_\alpha\left(\left\{\frac{ka_1}{N}\right\}\right), \dots, \tau_\alpha\left(\left\{\frac{ka_s}{N}\right\}\right)\right] \tau'_\alpha\left(\left\{\frac{ka_1}{N}\right\}\right) \cdots \tau'_\alpha\left(\left\{\frac{ka_s}{N}\right\}\right) - \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \cdots dx_s\right.$$

Proof. In the integral

$$\int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \cdots dx_s$$

we make the substitution $x_i = \tau_\alpha(z_i)$. The integrand after this substitution obviously turns out to belong to the class $E_s(\alpha)$, and therefore the result being proved follows from the corresponding theorems (^{1},^{2}).

$*$ $\{x\}$ is the fractional part of x.

Remark. Let $f(x_1, \dots, x_s)$ have a singularity on the hyperplane $x_1 = x_1^0, \dots, x_\nu = x_\nu^0$; $1 \leqslant \nu \leqslant s$;

$$\frac{\partial^{\alpha\nu} f(x_1, \dots, x_s)}{\partial x_1^\alpha \cdots \partial x_\nu^\alpha} = O\left(\frac{1}{r^{\beta+\alpha\nu}}\right), \quad 0 < \beta < \nu,$$

$$r = \sqrt{(x_1 - x_1^0)^2 + \dots + (x_\nu - x_\nu^0)^2}.$$

Suppose that f has the mixed derivatives occurring in the definition of the class $H_s(\alpha, c)$, continuous in the remaining part of the unit cube. Then one can compute integrals of such functions by introducing $\tau_{\alpha,i}(z)$ satisfying the condition

$$\tau_{\alpha,i}^{(k)}(z_i^0) = 0, \quad 1 \leqslant k \leqslant \alpha;$$

$$\tau_{\alpha,i}^{(\alpha)}(z_i) = O((z_i - z_i^0)^{b-\alpha}), \quad b = \frac{\alpha\nu}{\nu - \beta}, \quad \tau_{\alpha,i}(z_i^0) = x_i^0, \quad 1 \leqslant i \leqslant \nu \, * .$$

2. Let $f(x_1, \dots, x_s)$ be given on the unit cube and be different from zero on some parallelepiped whose faces are parallel to the coordinate planes. Suppose that $f(x_1, \dots, x_s)$ on this parallelepiped satisfies the conditions defining the function class $H_s(1, c)$. Extend $f(x_1, \dots, x_s)$ to the whole space by setting

$$f(x_1, \dots, x_s) = f(\{x_1\}, \dots, \{x_s\}).$$

Introduce

$$f_\Delta(x_1, \dots, x_s) = \int_{-\Delta}^{\Delta} \cdots \int_{-\Delta}^{\Delta} f[(x_1 + \xi_1), \dots, (x_s + \xi_s)]\, d\xi_1 \cdots d\xi_s,$$

$$\int_0^1 \cdots \int_0^1 f_\Delta(x_1, \dots, x_s)\, dx_1 \cdots dx_s = \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \cdots dx_s. \quad (1)$$

If $c(m_1, \dots, m_s)$ and $c_\Delta(m_1, \dots, m_s)$ are the Fourier coefficients of the functions $f(x_1, \dots, x_s)$ and $f_\Delta(x_1, \dots, x_s)$, then, obviously,

$$|c(m_1, \dots, m_s)| \leqslant \frac{c'}{m_1 \cdots m_s}, \quad (2)$$

$$|c_\Delta(m_1, \dots, m_s)| \leqslant \frac{c'}{m_1 \cdots m_s} \left| \frac{\sin 2\pi m_1 \Delta}{2\pi m_1 \Delta} \right| \cdots \left| \frac{\sin 2\pi m_s \Delta}{2\pi m_s \Delta} \right|, \quad (3)$$

$$c' = c'(c, s).$$

Consider

$$R^N(z, f) = \left| \frac{1}{N} \sum_{k=1}^{N} f\left(\frac{k}{N}, \frac{kz}{N}, \dots, \frac{kz^{s-1}}{N} \right) - \int_0^1 \cdots \int_0^1 f(x_1, \dots, x_s)\, dx_1 \cdots dx_s \right|; \quad (4)$$

$$R_\Delta^N(z, f) = \left| \frac{1}{N} \sum_{k=1}^{N} f_\Delta\left(\frac{k}{N}, \frac{kz}{N}, \dots, \frac{kz^{s-1}}{N} \right) - \right.$$

$$\left. - \int_0^1 \cdots \int_0^1 f_\Delta(x_1, \dots, x_s)\, dx_1 \cdots dx_s \right|; \quad (5)$$

$$r_\Delta(z, f) = \left| \frac{1}{N} \sum_{k=1}^{N} f\left(\frac{k}{N}, \frac{kz}{N}, \dots, \frac{kz^{s-1}}{N} \right) - \frac{1}{N} \sum_{k=1}^{N} f_\Delta\left(\frac{k}{N}, \frac{kz}{N}, \dots, \frac{kz^{s-1}}{N} \right) \right|. \quad (6)$$

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* The possibility of computing integrals of functions having singularities by means of $\tau_\alpha(z)$ was indicated by participants in the seminar on the number-theoretic Monte Carlo method at the V. A. Steklov Mathematical Institute of the Academy of Sciences of the USSR.

Theorem 2. If $N = p > s$ is prime, then there exists an integer $a = a(p)$, $1 \le a \le p-1$, such that

$$R^N(a, f) = O\left(\frac{\ln^s N}{N}\right).$$

Proof. From (1), (4), (5), and (6) we obtain

$$R^N(z, f) \le R^N_\Delta(z, f) + r_\Delta(z, f). \tag{7}$$

Let us estimate

$$\min_{1 \le z \le p-1} \max_{f \in H_s(\alpha, c)} R^p_\Delta(z, f).$$

Using the known estimates (4) and (3), we obtain

$$\min_{1 \le z \le p-1} \max_{f \in H_s(\alpha, c)} R^p_\Delta(z, f) \le \max_{f \in H_s(\alpha, c)} \frac{1}{p-1} \sum_{z=1}^{p-1} R^p_\Delta(z, f) \le$$

$$\le \frac{c'}{p-1} \sum_{m_1, \dots, m_s = -\infty}^{\infty}{}' \frac{1}{\overline{m}_1 \cdots \overline{m}_s} \left|\frac{\sin 2\pi m_1 \Delta}{2\pi m_1 \Delta}\right| \cdots \left|\frac{\sin 2\pi m_s \Delta}{2\pi m_s \Delta}\right| A(m_1, \dots, m_s),$$

where

$$0 \le A(m_1, \dots, m_s) \begin{cases} = p-1 & \text{if } d \equiv 0 \pmod{p}, \\ \le s-1 & \text{if } d \not\equiv 0 \pmod{p}; \end{cases}$$

d is the greatest common divisor of $m_1, \dots, m_s$. The prime on the summation sign means that the value $m_1 = \dots = m_s = 0$ is excluded. Take $\Delta = \frac{1}{p}$. Then the terms corresponding to tuples $m_1, \dots, m_s$ for which $d \equiv 0 \pmod{p}$ vanish if at least one $m_i \ne 0$. Consequently, for all terms different from zero, $A(m_1, \dots, m_s) \le s-1$. Thus,

$$\min_{1 \le z \le p-1} \max_{f \in H_s(\alpha, c)} R^p_\Delta(z, f) \le \frac{(s-1)c'}{p-1} \left(1 + 2 \sum_{m=1}^{\infty} \frac{1}{m} \left|\frac{\sin 2\pi m \Delta}{2\pi m \Delta}\right|\right)^s;$$

$$\left|\frac{\sin 2\pi m \Delta}{2\pi m \Delta}\right| \le \begin{cases} 1, & \text{if } m < \frac{1}{\Delta} = p, \\ \frac{1}{m\Delta}, & \text{if } m \ge \frac{1}{\Delta} = p. \end{cases}$$

Consequently,

$$\min_{1 \le z \le p-1} \max_{f \in H_s(\alpha, c)} R^p_\Delta(z, f) \le$$

$$\le \frac{(s-1)c'}{p-1} \left(1 + 2 \sum_{m=1}^{p-1} \frac{1}{m} + 2 \sum_{m=p}^{\infty} \frac{1}{m^2 \Delta}\right)^s = O\left(\frac{\ln^s p}{p}\right). \tag{8}$$

Let us now estimate $r_\Delta(z,f)$. If the point

$$\left(\frac{k}{p}, \left\{\frac{kz}{p}\right\}, \dots, \left\{\frac{kz^{s-1}}{p}\right\}\right)$$

is at a distance greater than Δ from the surface of the parallelepiped, then

$$\left| f\left(\frac{k}{p}, \frac{kz}{p}, \dots, \frac{kz^{s-1}}{p}\right) - f_\Delta\left(\frac{k}{p}, \frac{kz}{p}, \dots, \frac{kz^{s-1}}{p}\right)\right| \le cs\Delta.$$

The number of points

$$\left(\frac{k}{p}, \left\{\frac{kz}{p}\right\}, \dots, \left\{\frac{kz^{s-1}}{p}\right\}\right)$$

that are at a distance not exceeding Δ from the surface of the parallelepiped, for each z, does not exceed $\Delta p \cdot 2^s$. From what has been said there follows the following estimate for $r_\Delta(z,f)$:

$$r_\Delta(z,f) \le \Delta(cs + 2^s). \tag{9}$$

From (7), (8), (9) we obtain

$$\min_{1\le z\le p-1} \max_{f\in H_s(\alpha,c)} R^p(z,f) = O\left(\frac{\ln^s p}{p}\right),$$

which proves Theorem 2.

Remark. It is not difficult to show that

$$R^p(a_1,\dots,a_s) = \left|\frac{1}{p}\sum_{k=1}^{p} f\left(\frac{ka_1}{p},\dots,\frac{ka_s}{p}\right) - \right.$$

$$\left. - \int_0^1 \cdots \int_0^1 f(x_1,\dots,x_s)\, dx_1 \cdots dx_s \right| = O\left(\frac{\ln^s p}{p}\right),$$

where $a_1,\dots,a_s$ are the optimal coefficients determined in [2].

3. Let the equation be given

$$\varphi(x_1,\dots,x_s) = \int_0^{x_1} \cdots \int_0^{x_l} \int_0^1 \cdots \int_0^1 K(y_1,\dots,y_s,\ x_1,\dots,x_s)\, \varphi(y_1,\dots,y_s) \times$$

$$\times\, dx_1 \cdots dy_s + f(x_1,\dots,x_s), \qquad l \ge 1, \tag{10}$$

where $K(y_1, \dots, y_s, x_1, \dots, x_s) \in H_{2s}(\alpha, c_1)$, $f(x_1, \dots, x_s) \in H_s(\alpha, c_2)$. It is obvious that $\varphi(x_1, \dots, x_s) \in H_s(\alpha, c_3)$, $c_3 = c_3(c_1, c_2, s)$.

If $K_1(y_1, \dots, y_s, x_1, \dots, x_s) = K(y_1, \dots, y_s, x_1, \dots, x_s)$ for $0 \le y_i \le x_i$, $1 \le i \le l$, and $K_1(y_1, \dots, y_s, x_1, \dots, x_s) = 0$ at the remaining points of the unit cube, then (in the notation adopted) equation (10) can be rewritten in the form

$$\varphi(P) = \int_{G_s} K_1(Q, P)\varphi(Q)\,dQ + f(P). \qquad (11)$$

Take in the unit s-dimensional cube G_s the points

$$M_i = M\left(\frac{i}{N}, \left\{\frac{ia}{N}\right\}, \dots, \left\{\frac{ia^{s-1}}{N}\right\}\right),$$

where a is defined in Theorem 2.

Theorem 3. *If the quantities* $\widetilde{\varphi}(M_j)$ *satisfy the system of linear equations*

$$\widetilde{\varphi}(M_j) = \frac{1}{N}\sum_{i=1}^{N} K_1(M_i, M_j)\,\widetilde{\varphi}(M_i) + f(M_j) \qquad (j = 1, 2, \dots, N),$$

then the equality

$$\varphi(M_j) = \widetilde{\varphi}(M_j) + O\left(\frac{\ln^s N}{N}\right)$$

holds.

The proof of this theorem follows obviously from Theorem 2 and from [3].

The questions of the approximate solution of Volterra-type integral equations were also studied by Yu. N. Shakhov [5].

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CITED LITERATURE

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4. N. M. Korobov, *Vestn. MGU*, No. 4 (1959).
5. Yu. N. Shakhov, *DAN*, 128, No. 6 (1959).

Note: Figure translations are in progress. See original paper for figures.

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