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Abstract

Full Text

MATHEMATICS

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ON THE SMOOTHNESS OF GENERALIZED SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

(Presented by Academician P. S. Aleksandrov on 9 XII 1959)

1. In ⁽¹⁾ the linearized stationary hydrodynamic problem was studied

$$-\Delta \mathbf{U}(x) = \text{grad } p(x) + \mathbf{f}(x), \quad \text{div } \mathbf{U}(x) = 0 \quad \text{for } x \in \Omega;$$

$$\mathbf{U}(x) = 0 \quad \text{for } x \in \Gamma. \quad (1)$$

Here Ω is a bounded open domain with sufficiently smooth boundary Γ .

If the force $\mathbf{f}(x)$ satisfies a certain Hölder condition, then the formulas

$$u_i(x) = \int_{\Omega} G_{ij}(x, z) f_j(z) dz, \quad p(x) = \int_{\Omega} g_j(x, z) f_j(z) dz \quad (2)$$

give a classical solution of this problem, i.e. the functions $\mathbf{U}(x)$ and $p(x)$ are continuous in the closed domain $\bar{\Omega}$ and have in Ω as many continuous derivatives as enter into equations (1), satisfy these equations, and $\mathbf{U}(x) = 0$ on Γ ⁽¹⁾.

Let the function $p_1(x) = p(x)$ on Γ and $\Delta p_1 = 0$ in Ω .

Theorem 1. If \mathbf{f} is continuously differentiable, then the formula

$$p = -\Delta^{-1} \text{div } \mathbf{f} + p_1 \quad (3)$$

is valid.

From (3), for example, it follows that

$$\|\text{grad } p\|_{L_2(\Omega)} \leq C(q) \|\mathbf{f}\|_{L_q(\Omega)} \quad (4)$$

for any $q > 2$. This is proved with the aid of estimates from ^(1,2). From (4) and ⁽³⁾ it then follows that

$$\|\mathbf{U}\|_{W_2^2(\Omega)} \leq C_1(q) \|\mathbf{f}\|_{L_q(\Omega)}. \quad (5)$$

With the aid of (4) and (5) one establishes

Theorem 2. Let $\mathbf{f} \in L_q(\Omega)$, $q > 2$. Then formulas (2) give a generalized solution of problem (1); the equations are satisfied almost everywhere; $\mathbf{U} \in \dot{W}_2^2(\Omega)$, $p \in W_2^1(\Omega)$; the inequalities (4) and (5) are valid. If $q > 3$, then p and $\partial\mathbf{U}/\partial x_i$ are continuous in $\bar{\Omega}$.

2. Consider the nonlinear nonstationary problem

$$\frac{\partial \mathbf{U}}{\partial t} - \nu \Delta \mathbf{U} + u_k \frac{\partial \mathbf{U}}{\partial x_k} = \text{grad } p + \mathbf{f}, \quad \text{div } \mathbf{U} = 0 \quad \text{for } x \in \Omega, t > 0;$$

$$\mathbf{U}(t, x) = 0 \quad \text{for } x \in \Gamma, t \geq 0; \quad \mathbf{U}(0, x) = \mathbf{U}_0(x) \quad \text{for } x \in \Omega. \quad (6)$$

One type of its generalized solutions was studied by us in (4).

We shall first prove the existence of a solution under restrictions on the initial velocity \mathbf{U}_0 that are weaker than in (4). Then we shall trace how, for $t > 0$, the smoothness of the solution increases in connection with an increase in the smoothness of \mathbf{f} . In doing so, we shall make essential use of properties of fractional powers of certain self-adjoint operators. We note that a number of properties of fractional powers of operators were first discovered by M. A. Krasnosel'skii and used in the study of various equations (see, for example, (5)). An important role in our constructions is played by the work of V. P. Glushko and S. G. Krein (6).

To investigate (6), we pass (see (4)) to the integral equation

$$\begin{aligned} \mathbf{U}(t) = & \exp(-t\nu A) \mathbf{U}_0 - \int_0^t \exp[(s-t)\nu A] P \frac{\partial}{\partial x_k} (u_k \mathbf{U}) ds + \\ & + \int_0^t \exp[(s-t)\nu A] P \mathbf{f} ds. \end{aligned} \quad (7)$$

Making the substitution $\mathbf{V}(t) = t^{\alpha-\beta} A^\alpha \mathbf{U}(t)$, we obtain

$$\begin{aligned} \mathbf{V}(t) = & t^{\alpha-\beta} A^{\alpha-\beta} \exp(-t\nu A) A^\beta \mathbf{U}_0 - \\ & - t^{\alpha-\beta} \int_0^t A^{\alpha+1/2} \exp[(s-t)\nu A] A^{-1/2} P \frac{\partial}{\partial x_k} (A^{-\alpha} v_k \cdot A^{-\alpha} \mathbf{V}) S^{2(\beta-\alpha)} ds + \end{aligned}$$

$$+t^{\alpha-\beta}A^\alpha \int_0^t \exp[(s-t)\nu A]P\mathbf{f}ds. \quad (8)$$

Choose $\alpha = 1/4$ if the dimension of the space is $m = 2$, and $\alpha \in (3/8, 1/2)$ if $m = 3$. Let $\beta \in (2\alpha - 1/2, \alpha)$. We note that β may be chosen arbitrarily small if $m = 2$, and β must be greater than $1/4$ if $m = 3$. Let $\mathbf{U}_0 \in D(A^\beta)$, and let $\|\mathbf{f}(t)\|_{L_2(\Omega)}$, as a function of t , belong to $L_{\frac{1}{1-\alpha}+\varepsilon}$ for some $\varepsilon > 0$. Then, by the method of successive approximations, one can construct a unique continuous solution of equation (8), defined on some segment $[0, t_0]$. The proof of this fact is based on the properties of the operators $A^{-\alpha}$ established in ⁽⁴⁾. We define the desired solution by the formula $\mathbf{U}(t) = t^{\beta-\alpha}A^{-\alpha}\mathbf{V}(t)$.

Let $m = 2$. Then it follows from (7) and (8) that, for all $t \in [0, t_0]$, the function $A^\beta\mathbf{U}(t)$ is continuous; for $t \in (0, t_0]$ and $\gamma \in \left(\beta, \frac{\alpha + \varepsilon(1-\alpha)}{1 + \varepsilon(1-\alpha)}\right)$, the function $A^\gamma\mathbf{U}(t)$ is continuous, and the function $A^{\gamma-1}\mathbf{U}(t)$ is absolutely continuous and

$$\frac{\partial}{\partial t}(A^{\gamma-1}\mathbf{U}) + A^\gamma\mathbf{U} + \overline{A^{\gamma-1}P} \frac{\partial}{\partial x_k}(u_k\mathbf{U}) = \overline{A^{\gamma-1}P}\mathbf{f} \quad (\beta < \gamma < 1/2),$$

$$\frac{\partial}{\partial t}(A^{\gamma-1}\mathbf{U}) + A^\gamma\mathbf{U} + \overline{A^{\gamma-1}P} \left(u_k \frac{\partial \mathbf{U}}{\partial x_k} \right) = \overline{A^{\gamma-1}P}\mathbf{f} \quad (1/2 \leq \gamma < 1). \quad (9)$$

If $m = 3$, then by this method one succeeds in proving (9) only for $\gamma \leq 3/4$. In the case $m = 3$ we use the relation

$$u_i(t, x) = -\frac{1}{\nu} \int_\Omega u_k \frac{\partial u_i}{\partial x_k} G_{ij}(x, z) dz - \frac{1}{\nu} \int_\Omega \left\{ \left(\frac{\partial}{\partial x_k} A^{-3/4} \right) \frac{\partial}{\partial t} (A^{-1/4} u_j) \right\} \frac{\partial}{\partial x_k} G_{ij}(x, z) dz + \frac{1}{\nu} \int_\Omega f_j G_{ij}(x, z) dz. \quad (10)$$

A similar relation was established in ⁽⁷⁾. It makes it possible to study the smoothness of a solution with respect to the spatial variables. In doing so, estimates from ⁽¹⁾ and the following are used.

Lemma. The operator $\frac{\partial}{\partial x} A^{-(\alpha+\varepsilon)}$ ($1/2 \leq \alpha \leq 1$), for any $\varepsilon > 0$, maps H into $L_{\frac{6}{5-4\alpha}}(\Omega)$ and is bounded.

This lemma follows from ^(6,8) and (5). Thus, it is proved that $\|U(t)\|_{L_q(\Omega)}$, for any $q > 1$, belongs (in t) to the same L_r as $\|f(t)\|_{L_2(\Omega)}$. This makes it possible to establish (9) also for $m = 3$.

Now let $\|f(t) - f(s)\|_{L_2(\Omega)} \leq K|t - s|^\varepsilon$ for some $\varepsilon > 0$. Then from (7) and (10) it follows that, for any $\varepsilon_1 < \min(\varepsilon, 1/2)$, the function $A^{\varepsilon_1}U(t)$ is continuously differentiable with respect to t . Relation (10) then makes it possible to discover a new improvement of the differential properties of U : this function, in the norm $W_6^1(\Omega)$, satisfies in t the condition $\text{Lip } \varepsilon_1$. Consequently, $u_k \frac{\partial U}{\partial x_k}$ satisfies this condition in the norm $L_6(\Omega)$.

Suppose, further, that $\|f(t) - f(s)\|_{L_q(\Omega)} \leq K|t - s|^\varepsilon$ for some $q > 2$. Then from Theorem 2 it follows that the functions U and

$$p = \int_{\Omega} \left[f_j - \frac{\partial u_j}{\partial t} - u_k \frac{\partial u_j}{\partial x_k} \right] g_j(x, z) dz$$

satisfy equations (6) almost everywhere, and for $t > 0$ belong respectively to the spaces $W_2^2(\Omega)$ and $W_2^1(\Omega)$ and satisfy, in the metrics of these spaces, the condition $\text{Lip } \varepsilon_1$ in t . If $q > 3$ and $\varepsilon > 1/4$, then p and $\frac{\partial U}{\partial x_k}$ are continuous in $(0, t_0] \times \bar{\Omega}$.

Let additionally $\|f(t) - f(s)\|_{L_2(\Omega)} \leq K|t - s|^\eta$. Then, for any $\eta_1 < \eta$, the function $\frac{d}{dt}(A^{\eta_1}U)$ satisfies in the norm $L_2(\Omega)$ some Hölder condition. For the proof one must use (7) and take into account that the function $A^{1/2}P \left(u_k \frac{\partial U}{\partial x_k} \right)$ satisfies in the norm H the condition $\text{Lip } \varepsilon_1$ in t . If $\eta > 3/4$, then $\frac{\partial U}{\partial t}$ satisfies the ordinary Hölder condition in the joint variables t and x ^(6,8). Suppose that $f(t, x)$ also has this property. Then from (10) it follows that $u_k \frac{\partial U}{\partial x_k}$ also has the same property, and from ⁽¹⁾ it follows, for example:

Theorem 3. Suppose that the function $f(t, x)$ satisfies, in the joint variables, the condition $\text{Lip } \gamma$ for some $\gamma > 3/4$. Then U and p constitute a classical (for $t > 0$) solution of problem (6).

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