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Abstract

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MATHEMATICS

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ON STATIONARY GROUPS OF MOTIONS OF SPACES OF AFFINE CONNECTION

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Let A_n be a space of affine connection with base manifold M_n; let S be a group of its motions leaving fixed a certain point $O \in M_n$; and let E_n be the tangent vector space at the point O.

Join an arbitrary point $x \in M_n$ by a geodesic to the point O and consider the development ([1]) of this geodesic in E_n—the segment of the straight line Ox'. By assigning to the point $x \in M_n$ the point $x' \in E_n$, we define a one-to-one correspondence between certain neighborhoods of the base manifold and of the tangent space E_n. For convenience of exposition, in what follows we shall identify the corresponding points x and x'; lower-case Latin letters $x, y, u, \ldots$ will denote simultaneously vectors of E_n and the corresponding points of M_n. The components x^i of these vectors with respect to an arbitrary basis in E_n are the well-known normal coordinates in M_n. This same convention determines the meaning of the terms linear subspace, the point $y - x$ (the point of M_n corresponding to the difference of the vectors $y - x$ in E_n), etc., used below as applied to the base manifold. It is important to note that, by virtue of this same convention, one may regard the group S as a group of nonsingular real matrices of order n.

We shall also introduce the following definitions. If S is a linear (matrix) group, then by $\overline{S}$ we shall denote the semigroup—the closure of S in the space of all matrices of the given order* . Further, let E_n be a vector space in which a certain linear group S acts, and let E_n^k be the space whose points are systems of k linearly independent vectors from E_n (i.e., the direct product of k copies of E_n, from which the cone corresponding to systems of linearly dependent vectors has been removed). We shall call the linear group S k-transitive if it is transitive ([2]) as a group of transformations of E_n^k (of each connected component of E_n^k in the case $k = n$). We shall call a system of k vectors (or a point of E_n^k) an exact frame if there exists no nontrivial transformation of the group S leaving this system fixed, and if no part of this system satisfies the latter condition. If almost all systems of k vectors from E_n (i.e., all points of E_n^k with the exception, possibly, of a set of measure zero–surfaces in E_n^k) are exact frames, then we shall say that a frame of the group contains k vectors.

Consider arbitrary points u, x of M_n (or vectors of E_n). Construct the development of the geodesic broken line Oux, consisting of the geodesic arcs Ou and ux. This development is a broken line, one of whose segments coincides with Ou (by virtue of the identification of the points of M_n and E_n adopted above), while the other is a certain segment uy. This construction makes it possible to define a vector-function of two vector arguments $f(u, x)$ by means of the formula

$$f(u, x) = y - x. \tag{1}$$

* $\overline{S}$, generally speaking, also contains singular matrices.

It is easy to see that $f(u, x)$ is uniquely determined for any pair u, x from some neighborhood of the point O containing no geodesic triangles. Further, if A_n belongs to the class C_{p+1}, then $f(u, x)$ is at least p times continuously differentiable in all arguments; we shall assume that $p \geq 2$.

Below we use the following properties of $f(u, x)$:

1) If $u = \lambda x$ (λ is a scalar), then $f(u, x) = 0$.

Indeed, in this case the arcs Ou and ux form one smooth geodesic arc, whose development coincides with it, whence $f(u, x) = x - x = 0$. In particular, $f(u, 0) = f(0, x) = 0$.

2) If $\alpha \in S$, then

$$f(\alpha u, \alpha x) = \alpha f(u, x). \tag{2}$$

Indeed, let γ be an arbitrary arc with initial point at O, and let $R(\gamma)$ be its development; let the transformation α carry the arcs γ and $R(\gamma)$, respectively, into $\alpha\gamma$ and $\alpha R(\gamma)$. If $\alpha \in S$, then the transformation α (a motion) preserves the development operation, i.e. $\alpha R(\gamma) = R(\alpha\gamma)$, where $R(\alpha\gamma)$ is the development of the arc $\alpha\gamma$. If, in particular, γ is the broken line O, u, x, $R(\gamma)$ is $O, u, (f(u, x) + x)$ (according to (1)), then $\alpha\gamma$ coincides with the broken line $O, \alpha u, \alpha x$, and $R(\alpha\gamma)$ with the broken line $O, \alpha u, (f(\alpha u, \alpha x) + \alpha x)$; finally, $\alpha R(\gamma)$ with the broken line $O, \alpha u, \alpha(f(u, x) + x)$, whence we obtain (2).

3) If H is a linear subspace and $f(h_1, h_2) \in H$ for any h_1 and h_2 from H, then H (as a submanifold in M_n) is a totally geodesic surface.

4) If $f(u, x)$ is linear in x and A_n is a torsion-free connection, then A_n is a locally affine connection.

Theorem 1. Let $\alpha \in \bar{S}$; let U_k, V_{n-k} be complementary linear subspaces such that $\alpha^p U_k = 0$, $\alpha^p E_n = V_{n-k}$ for some integer p. Then U_k, V_{n-k} are totally geodesic surfaces. Every linear subspace of dimension $k+1$ containing U_k is also a totally geodesic surface.

We prove one of the assertions of the theorem. Let $v_1 \in V_{n-k}$, $v_2 \in V_{n-k}$. Then $v_1 = \alpha^p x_1$, $v_2 = \alpha^p x_2$, where x_1, x_2 are some vectors from E_n. Further,

$$f(v_1, v_2) = f(\alpha^p x_1, \alpha^p x_2) = \alpha^p f(x_1, x_2) \in V_{n-k}.$$

From property 3) it now follows that V_{n-k} is a totally geodesic surface. The remaining assertions of the theorem are proved in a similar way.

As an example of the application of the same method, let us examine in detail one special case. Namely, let S contain the group of all linear transformations leaving fixed $(n-1)$ certain vectors. Let V_{n-1} be the linear subspace spanned by these vectors. For any x not belonging to V_{n-1}, one can find a matrix $\alpha_x \in \bar{S}$ such that $\alpha_x x = 0$, $\alpha_x E_n = V_{n-1}$. If x, y are arbitrary vectors from E_n, and α_x, α_y are the corresponding matrices from $\bar{S}$, then

$$\alpha_x f(x, y) = f(\alpha_x x, \alpha_x y) = f(0, \alpha_x y) = 0;$$

in exactly the same way, $\alpha_y f(x, y) = 0$. As is easy to see, it follows that $f(x, y) = 0$. Thus, if S contains the indicated subgroup and A_n is a torsion-free connection, then A_n is a locally affine connection.

Theorems 2 and 3 are proved by this same method.

Theorem 2. For any sequence $\{\alpha_k\} \in S$ $(k = 1, 2, \dots, \infty)$, the following linear subspaces are totally geodesic surfaces: a) the maximal U possessing the property that $\alpha_k U \to 0$; b) the maximal V such that the sequence $\{\alpha_k x\}$ is bounded for any $x \in V$.

For the proof it suffices to consider the sequence $\{\alpha_k f(x_1, x_2)\}$ $(k = 1, 2, \dots, \infty)$, where x_1 and x_2 are vectors from the subspace U (or V). By properties 1), 2), this sequence of the vector-function $f(x_1, x_2)$ converges to 0 (or is bounded), and therefore $f(x_1, x_2)$ belongs-

belong to the subspace U (respectively, V). From property 3) it now follows that U (V) is a totally geodesic surface.

Applying Theorem 2 (with minor additions) to the sequence of powers of the matrix, we obtain Theorem 3.

Theorem 3. *For any real number c $(0 < c < 1)$, the invariant subspace corresponding to all eigenvalues of matrices $a \in S$ whose absolute value does not exceed c is a totally geodesic surface.*

Of course, the theorem is also valid for eigenvalues not less than some $c > 1$.

Theorem 4. *If S is doubly transitive and A_n is a connection without torsion, then A_n is a locally affine connection.*

For the proof we represent $f(u, x)$ in the form

$$f(u,x) = l(u,x) + m(u,x),$$

where $l(u,x)$ is linear with respect to x, and

$$\frac{|m(u,x)|}{|x|} \to 0 \qquad \text{as } x \to 0.$$

(By the symbol $|x|$ we denote the norm of the vector, defined, for example, as follows:

$$|x| = \left(\sum_i x^{i2}\right)^{1/2},$$

where x^i are the components of x in some fixed basis. As the norm of the matrix a we shall, as usual, take

$$|a| = \max \frac{|ax|}{|x|}.$$

If $a \in S$, then $m(au, ax) = am(u,x)$. On the other hand, for any u and x from M_n one can find a sequence $\{a_k\} \in S$ such that $a_k u \to 0$, $a_k x \to 0$, and $|a_k x| \cdot |a_k^{-1}| < c$; then

$$|m(u,x)| \leq |a_k x| \cdot |a_k^{-1}| \cdot \frac{|m(a_k u, a_k x)|}{|a_k x|} \to 0,$$

i.e. $f(u,x)$ is linear with respect to x.

Let us also note two simple theorems.

Theorem 5. *If the frame of the group S contains n vectors, then A_n is a subprojective connection.*

Theorem 6. *If the frame of the group S contains more than two vectors, then every two-dimensional linear manifold ("plane" passing through the point O) belongs to some proper linear subspace $\bar{H}_k$ $(k < n)$, which is a totally geodesic surface.*

Indeed, let U_2 be an arbitrary plane in E_n; consider U_{2+k}—the linear span of U_2 and all values of the vector-function f from all possible pairs of vectors in U_2, then U_{2+k+l}—the linear span of U_{2+k} and the values of f from pairs of vectors in U_{2+k}, and so on. If this process terminates in some proper linear subspace, then it is a totally geodesic manifold (according to 3)), containing U_2. Otherwise, as is easy to see, any pair of vectors from U_2 is a frame of the group S.

Theorem 5 is proved in the same way.

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CITED LITERATURE

1. É. Cartan, *The Geometry of Riemannian Spaces*, Moscow, 1948.

2. N. G. Chebotarev, *Theory of Lie Groups*, Moscow—Leningrad, 1940.

Note: Figure translations are in progress. See original paper for figures.