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Abstract

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ON A METHOD FOR SOLVING THE GENERAL STEFAN PROBLEM

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The well-known Stefan problem on the distribution of heat during the freezing of a substance, under various particular assumptions, has been studied in a number of works. The most general results were obtained in (^{1,2}). In the present note the solution of the general Stefan problem is obtained as the limit of solutions of boundary-value problems for certain quasilinear equations with smooth coefficients, analogously to how in (^{3}) boundary-value problems for linear second-order equations with discontinuous coefficients were solved. We consider the case of an arbitrary finite number of phases, assuming that the coefficients of the parabolic equations describing the distribution of heat in each phase, and the coefficients in the condition on the line of separation of two phases, are variable. In the same way one can solve the stationary Stefan problem, which leads to the solution of elliptic equations with an unknown boundary, and the corresponding problem in space (^{4}).

1. **Statement of the problem.** Let S_i $(i = 1, \dots, N)$ be given constants (values of the temperature u at which changes of the phase state of the substance occur). The points S_i $(S_i < S_{i+1})$ divide the u-axis into $N + 1$ intervals, which we number in the order of increasing u. We denote by l_i the i-th interval.

By a solution of the Stefan problem we shall mean a continuous function $u(t, x)$, defined in the rectangle $Q\{0 \leq t \leq T,\ 0 \leq x \leq l\}$, and satisfying the following conditions:

1) At the interior points of Q where the value $u(t, x)$ belongs to l_i, the function $u(t, x)$ satisfies the equation

$$a_i(t,x)\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + b_i(t,x)\frac{\partial u}{\partial x} + c_i(t,x)u + f_i(t,x), \qquad i = 1, \dots, N, \quad (1)$$

where a_i, b_i, c_i, f_i are given smooth functions in Q, $a_i \geq \bar{a} > 0$ ($\bar{a}$ is some constant).

2) On the lines $x = x_i(t)$, where $u(t, x) = S_i$, the condition

$$k_i(t,x)\left.\frac{\partial u}{\partial x}\right|_{u=S_i-0} - k_{i+1}(t,x)\left.\frac{\partial u}{\partial x}\right|_{u=S_i+0} = \lambda_i(t,x)\frac{dx_i}{dt}, \quad (2)$$

is satisfied, where $k_i(t,x) \geq \bar{k} > 0$, $\lambda_i(t,x) \geq 0$ are smooth functions given in Q, $i = 1, \dots, N$.

3)

$$u(0,x) = u_0(x), \qquad u(t,0) = u_1(t), \qquad u(t,l) = u_2(t), \tag{3}$$

where $u_0(x)$, $u_1(t)$, $u_2(t)$ are given functions.

A function $u(t,x)$ with the indicated properties will be called a classical solution of the Stefan problem. Below we shall define a generalized solution of the Stefan problem, prove its existence and uniqueness, and investigate the question of the fulfillment for it of conditions 1)–3).

2. **Definition of a generalized solution of the Stefan problem.** We shall show that the classical solution of the Stefan problem (1)–(3)

satisfies a certain integral identity, which we shall use as the basis for the definition of a generalized solution of problem (1)–(3). Denote by $k(t,x,u)$, $a(t,x,u)$, $b(t,x,u)$, $c(t,x,u)$, $f(t,x,u)$ the functions equal respectively to the functions $k_i(t,x)$, $a_i(t,x)$, $b_i(t,x)$, $c_i(t,x)$, $f_i(t,x)$ for u from l_i. We multiply equation (2) by k_i and write it in the form

$$a_i k_i \frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(k_i \frac{\partial u}{\partial x}\right) + \left(k_i b_i - \frac{\partial k_i}{\partial x}\right)\frac{\partial u}{\partial x} + k_i c_i u + k_i f_i. \tag{4}$$

Let

$$\varphi(t,x,u) = \int_{c_0}^{u} a(t,x,s)k(t,x,s)\,ds + \gamma_i$$

for u from l_i, where

$$\gamma_i(t,x) = \sum_{j=1}^{i-1} \lambda_j(t,x),$$

c_0 is some constant, $c_0 < S_1$. It is obvious that

$$\varphi(t,x,S_i+0) - \varphi(t,x,S_i-0) = \lambda_i(t,x).$$

We shall denote by h_x the partial derivative with respect to x of $h(t,x,u)$, and by $\partial h/\partial x$ the derivative with respect to x of $h(t,x,u(t,x))$. The notations h_t and $\partial h/\partial t$ are used analogously. Let

$$\psi(t,x,u) = \int_{c_0}^{u} (kb - k_x)\, ds, \qquad \vartheta(t,x,u) = \int_{c_0}^{u} k(t,x,s)\, ds.$$

We write equation (4) in the form

$$\frac{\partial \varphi}{\partial t} = \frac{\partial^2 \vartheta}{\partial x^2} + \frac{\partial(\psi - \vartheta_x)}{\partial x} + H, \tag{5}$$

where

$$H(t,x,u) = \varphi_t - \psi_x + kcu + kf.$$

Let $F(t,x)$ be a smooth function, $F(T,x) = F(t,0) = F(t,l) = 0$. Multiplying (5) by $F(t,x)$, integrating over Q, and transforming separate terms by integration by parts, we obtain

$$\iint_Q \left[\varphi(t,x,u)\frac{\partial F}{\partial t} + \vartheta(t,x,u)\frac{\partial^2 F}{\partial x^2} - (\psi(t,x,u) - \vartheta_x(t,x,u))\frac{\partial F}{\partial x} + H(t,x,u)F\right] dQ + \int_0^l F(0,x)\varphi(0,x,u_0(x))\, dx$$

$$- \int_0^T \vartheta(t,0,u_1(t))\frac{\partial F}{\partial x}\, dt + \int_0^T \vartheta(t,l,u_2(t))\frac{\partial F}{\partial x}\, dt = 0, \tag{6}$$

since the integrals along the lines $x = x_i(t)$, where $S_i = u$, are equal to zero by virtue of condition (2).

By a generalized solution of the Stefan problem (1)–(3) we shall mean a bounded summable function $u(t,x)$ which satisfies the integral identity (6) for every $F(t,x)$ from the indicated class. At the same time, for each generalized solution $u(t,x)$, the functions $\varphi(t,x,u)$ and $H(t,x,u)$, which in general are discontinuous at $u = S_i$, are assigned, for $u(t,x) = S_i$, certain bounded values, with

$$\varphi(t,x,S_i - 0) \leq \varphi(t,x,S_i) \leq \varphi(t,x,S_i + 0).$$

3. Uniqueness theorem. We shall show that the bounded function $u(t,x)$ satisfying (6) is unique. This proof is close in method to that carried out in [5]. We shall assume that $\lambda_i(t,x) \equiv \lambda_i(t)$, $i = 1,\dots,N$, $c(t,x,u) \equiv c(t,x)$, $f(t,x,u) \equiv f(t,x)$, i.e. the function $H(t,x,u)$ is continuous in u. The proof remains valid also in the general case if $\lambda_i(t,x) \geq \bar{\lambda} > 0$ and the set of points where $u = S_i$ has measure zero.

Let $u_1(t,x)$ and $u_2(t,x)$ be two generalized solutions of problem (1)–(3). We shall show that $u_1 \equiv u_2$.

Using (6), we have

$$\iint_Q [\varphi(t,x,u_1) - \varphi(t,x,u_2)] \left[\frac{\partial F}{\partial t} + \frac{v(t,x,u_1) - v(t,x,u_2)}{\varphi(t,x,u_1) - \varphi(t,x,u_2)} \frac{\partial^2 F}{\partial x^2} + \right.$$

$$\left. + \frac{\mu(t,x,u_1) - \mu(t,x,u_2)}{\varphi(t,x,u_1) - \varphi(t,x,u_2)} \frac{\partial F}{\partial x} + \frac{H(t,x,u_1) - H(t,x,u_2)}{\varphi(t,x,u_1) - \varphi(t,x,u_2)} F \right] dQ = 0, \qquad (7)$$

where $\mu = -\psi + v_x$. Note that $0 \leq (u_1 - u_2)/(\varphi(t,x,u_1) - \varphi(t,x,u_2)) \leq M$. Therefore the coefficients of $\partial^2 F/\partial x^2$ and $\partial F/\partial x$ in (7), and also, by virtue of the assumptions made, the coefficient of F, are bounded in Q. Denote them, respectively, by A, B, and C. We shall show that

$$\iint_Q [\varphi(t,x,u_1) - \varphi(t,x,u_2)] \Phi(t,x)\, dQ = 0, \qquad (8)$$

where $\Phi(t,x)$ is an arbitrary smooth finite function in Q. To this end, consider in Q the solution F_{nm} of the equation

$$\frac{\partial F_{nm}}{\partial t} + A_n \frac{\partial^2 F_{nm}}{\partial x^2} + B_n \frac{\partial F_{nm}}{\partial x} + C_{mF_{nm}} = \Phi \qquad (9)$$

with the conditions $F_{nm}(T,x) = 0$, $F_{nm}(t,0) = 0$, $F_{nm}(t,1) = 0$, where A_n, B_n, C_m are smooth functions, bounded for all m and n, such that A_n, B_n, C_m converge in mean as $n, m \to \infty$, respectively, to A, B, C. Moreover $A_n \geq 1/n$, $\|A_n - A\|$ in $L_2(Q)$ is less than α/n, where $\alpha > 0$ is some number, and $B_n = A_n \bar{B}_n$, while $\bar{B}_n$ converges in mean as $n \to \infty$ to the bounded function $(\mu(t,x,u_1) - \mu(t,x,u_2))/(v(t,x,u_1) - v(t,x,u_2))$. By the maximum principle, $|F_{nm}| \leq M_1$, where M_1 does not depend on m and n. Multiplying equation (9) by $e^{\alpha t} \partial^2 F_{nm}/\partial x^2$, integrating over Q, and transforming individual terms by integration by parts, we obtain

$$\iint_Q A_n \left(\frac{\partial^2 F_{nm}}{\partial x^2} \right)^2 dQ < M_2(m), \qquad \iint_Q \left(\frac{\partial F_{nm}}{\partial x} \right)^2 dQ < M_3(m) \qquad (10)$$

for all n. Substituting F_{nm} into (7) and using (9), we have

$$\iint_Q [\varphi(t,x,u_1) - \varphi(t,x,u_2)] \left[\Phi + (A - A_n) \frac{\partial^2 F_{nm}}{\partial x^2} + (B - B_n) \frac{\partial F_{nm}}{\partial x} + \right.$$

$$\left. + (C - C_m) F_{nm} \right] dQ = 0. \qquad (11)$$

It is easy to show that, for sufficiently large m and n, the modulus of the left-hand side of (8) is less than ε, where $\varepsilon > 0$ is arbitrary. Hence (8) follows, and the equality $u_1 \equiv u_2$. Indeed, choose m so large that the integral of the last term in (11) is less than $\varepsilon/3$. Then, for sufficiently large n, by virtue of the estimates (10), the integral of the penultimate term in (11) is also less than $\varepsilon/3$, and

$$\left|\iint_Q (A - A_n)\frac{\partial^2 F_{nm}}{\partial x^2}\,dQ\right| \leq \left[\iint_Q A_n\left(1 - \frac{A}{A_n}\right)^2 dQ\right]^{1/2}\left[\iint_Q A_n\left(\frac{\partial^2 F_{nm}}{\partial x^2}\right)^2 dQ\right]^{1/2} < \varepsilon/3, \tag{12}$$

since the first integral on the right-hand side of (12), taken over the set E, where $A_n < \sigma$, does not exceed $\sigma\alpha^2$, and over the set $Q - E$ does not exceed $\alpha^2/\sigma n^2$. The theorem is proved.

4. Construction of a solution of the Stefan problem (1)–(3). Let $k_i \subset C^{(2)}$, λ_i and the coefficients of (1) belong to the class $C^{(1)}$ in Q, $u_0(x) \subset C^{(1)}$, and let $v|_{x=0,\,u=u_1(t)}$, $v|_{x=1,\,u=u_2(t)}$ belong to $C^{(2)}$. Construct sequences of smooth functions v^n, ψ^n $(n = 1, \ldots)$, converging uniformly together with the derivatives v_x^n and ψ_x^n to v and ψ, respectively, the first derivatives of ψ^n and v^n and all second derivatives of v^n, except v_{uu}^n, being uniformly bounded with respect to n.

Let $\varphi^n, c^n, f^n, \varphi_t^n$ be bounded sequences of smooth functions, converging as $n \to \infty$, respectively, to φ, kc, kf, and φ_t uniformly on every closed set not containing the points $u = S_i$; $v_u^n \geq \chi > 0$, $\varphi_u^n \geq p > 0$.

Consider the parabolic equation, quasilinear with respect to v^n,

$$\frac{\partial \varphi^n}{\partial t} = \frac{\partial^2 v^n}{\partial x^2} + \frac{\partial(\psi^n - v_x^n)}{\partial x} + \varphi_t^n - \psi_x^n + c^n u^n + f^n \tag{13}$$

with the conditions

$$v^n\big|_{t=0} = v\big|_{t=0,\,u=u_0(x)}, \qquad v^n\big|_{x=0} = v\big|_{x=0,\,u=u_1(t)}, \qquad v^n\big|_{x=l} = v\big|_{x=l,\,u=u_2(t)}. \tag{14}$$

The solution $v^n = v^n(t, x, u^n)$ of problem (13), (14) was constructed in ([6]). We shall show that the generalized solution $u(t, x)$ of problem (1)–(3) is the limit, as $n \to \infty$, of a uniformly convergent sequence $u^n(t, x)$ of solutions of problem (14), (13). Equation (13) can be written with respect to u^n in the form of a parabolic equation for whose solutions the maximum principle is valid. Therefore $|u^n| < M_4$, where M_4 does not depend on n. We now estimate the derivatives of u^n. Let the continuously differentiable function $V(t, x)$ coincide with $\partial v^n/\partial t$ for $x = 0$ and for $x = l$. Multiply both sides of equation (13) by $e^{\theta t}(\partial v^n/\partial t - V)$ and integrate over $Q_\tau\{0 \leq t \leq \tau,\ 0 \leq x \leq l\}$. Taking into account on the left-hand side of this equality that $\partial v^n/\partial t = k_u \partial u^n/\partial t + v_t^n$, and transforming by integration by parts the integral of $\frac{\partial \varphi^n}{\partial t}(v_t - V)e^{\theta t}$ and the first integral on the right-hand side, we obtain the energy estimate

$$\iint_Q \left(\frac{\partial u^n}{\partial t}\right)^2 dQ + \iint_Q \left(\frac{\partial v^n}{\partial x}\right)^2 dQ + \int_0^l \left(\frac{\partial v^n(\tau, x)}{\partial x}\right)^2 dx < M_5, \tag{15}$$

where M_5 does not depend on n and τ. From (15) and the boundedness of u^n there follows the compactness of $\{u^n\}$ in the sense of uniform convergence. Choose a subsequence u^n converging to $u(t,x)$. It is obvious that the corresponding v^n also converge uniformly. We shall show that $u(t,x)$ satisfies (6). Multiplying (13) by $F(t,x)$, integrating over Q, and transforming individual terms by integration by parts, we obtain an identity for u^n analogous to (6).

Let $H^n = \varphi_t^n - \psi_x^n + c^n u^n + f^n$, and let the subsequence u^n be such that the corresponding φ^n and H^n converge weakly to certain functions $\widetilde{\varphi}$ and $\widetilde{H}$. It is obvious that at the points (t,x) where $u(t,x) \neq S_i$, $\widetilde{\varphi}(t,x) = \varphi(t,x,u(t,x))$ and $\widetilde{H}(t,x) = H(t,x,u(t,x))$. At the points where $u = S_i$, we set $\varphi = \widetilde{\varphi}$ and $H = \widetilde{H}$. It is easy to show that

$$\varphi(t,x,S_i - 0) \leq \varphi(t,x,S_i) \leq \varphi(t,x,S_i + 0).$$

Passing to the limit in the indicated identity for u^n along this subsequence, we obtain (6).

At the points (t,x) where $u(t,x) \neq S_i$, the function $u(t,x)$ satisfies (1) in the ordinary sense, since in a neighborhood of such a point it is a generalized solution of a linear parabolic equation with smooth coefficients. Condition (2) is fulfilled in the integral sense. By virtue of the uniqueness theorem, the entire sequence $u^n(t,x)$ converges uniformly to $u(t,x)$.

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