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Abstract

Full Text

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A LIMIT THEOREM FOR INDEPENDENT RANDOM VARIABLES

(Presented by Academician A. N. Kolmogorov on 16 IV 1960)

Let $\xi_1, \xi_2, \ldots, \xi_n, \ldots$ be a sequence of independent identically distributed random variables for which $M\xi_i = 0,\ D\xi_i = 1$. Put

$$S_{n0} = 0, \qquad S_{nk} = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \xi_i.$$

Further, let the functions $g_1(t)$ and $g_2(t)$ be defined for $t \in [0,1]$, with $g_1(0) < 0 < g_2(0)$, and for all $t \in [0,1]$ let $g_1(t) < g_2(t)$, and for some K, for all t_1 and t_2 in $[0,1]$, the inequality

$$|g_1(t_1) - g_1(t_2)| + |g_2(t_1) - g_2(t_2)| \leq K|t_1 - t_2|$$

holds.

Denote by Q_n the probability

$$Q_n = P\left\{g_1\left(\frac{k}{n}\right) < S_{nk} < g_2\left(\frac{k}{n}\right), \quad k = 0, 1, 2, \ldots, n\right\}.$$

Now consider the Brownian motion process $w(t)$, for which $Mw(t) = 0,\ Dw(t) = t$, and denote

$$Q = P\{g_1(t) < w(t) < g_2(t), \quad 0 \leq t \leq 1\}.$$

It is known (see, for example, (1), Ch. IV, § 2) that $Q_n \to Q$ as $n \to \infty$. Under certain additional conditions one can estimate the rate of convergence of Q_n to Q. However, even in the case of bounded ξ_i, i.e., under the condition that for some c

$$P\{|\xi_i| > c\} = 0, \tag{*}$$

up to the present time one could use only the estimate of Yu. V. Prokhorov ([2])

$$|Q_n - Q| = O(n^{-1/4} \log^2 n).$$

We prove, under assumption (*), the stronger result:

Theorem. There exists a constant H, depending only on K, C, $g_1(0)$, and $g_2(0)$, such that for all n

$$|Q - Q_n| \le H \frac{\log n}{\sqrt{n}}. \tag{1}$$

This theorem is also valid in the case where $\xi_1, \xi_2, \dots$ have different distributions, but for all i

$$M\xi_i = 0, \qquad D\xi_i = 1, \qquad P\{|\xi_i| > C\} = 0.$$

The proof of the theorem is based on the following lemma, which may prove useful in studying the convergence of a sequence of sums of independent random variables to a Brownian-motion process.

Lemma. If $\xi_1, \xi_2, \dots, \xi_n$ are independent random variables for which $\mathbf{M}\xi_i = 0$, then one can specify independent nonnegative quantities $\tau_1, \tau_2, \dots, \tau_n$ such that the quantities

$$w(\tau_1), \quad w(\tau_1 + \tau_2), \quad w(\tau_1 + \tau_2 + \dots + \tau_n)$$

($w(t)$ is a Brownian-motion process) have the same joint distribution as the quantities $\xi_1, \xi_1 + \xi_2, \xi_1 + \xi_2 + \dots + \xi_n$, and moreover:

a) if $\mathbf{D}\xi_i < \infty$, then $\mathbf{M}\tau_i = \mathbf{D}\xi_i$;

b) there exist constants L_m, $m > 0$, such that

$$\mathbf{M}\tau_i^m \leqslant L_m \mathbf{M}|\xi_i|^{2m};$$

c) if $|\xi_i| \leqslant C$, then

$$\sup_{0 \leqslant s \leqslant \tau_i} \left| w\left(\sum_{k=1}^{i-1} \tau_k + s\right) - w\left(\sum_{k=1}^{i-1} \tau_k\right) \right| \leqslant C;$$

d) if the ξ_i are identically distributed, then the τ_i are also identically distributed.

Using this lemma, instead of the quantities s_{nk} we may consider the quantities $w\left(\sum_{i=1}^{k}\tau_i^{(n)}\right)$, where $\tau_i^{(n)}$ are the quantities corresponding by the lemma to the quantities $\frac{1}{\sqrt{n}}\xi_i$. In this case $\mathbf{M}\tau_i^{(n)}=0$, $\mathbf{D}\tau_i^{(n)}=b^2/n^2$, where b^2 is a certain constant.

Putting

$$\frac{1}{\sqrt{n\mathbf{D}\tau_1^{(n)}}}\sum_{i=1}^{k}\left(\tau_i^{(n)}-\mathbf{M}\tau_i^{(n)}\right)=\frac{\sqrt{n}}{b}\left(\sum_{i=1}^{k}\tau_i^{(n)}-\frac{k}{n}\right)=\zeta_{nk},$$

we shall have

$$w\left(\sum_{i=1}^{k}\tau_i^{(n)}\right)=w\left(\frac{k}{n}+\frac{b}{\sqrt{n}}\zeta_{nk}\right).$$

Thus,

$$Q_n=\mathbf{P}\left\{g_1\left(\frac{k}{n}\right)<w\left(\frac{k}{n}+\frac{b}{\sqrt{n}}\zeta_{nk}\right)<g_2\left(\frac{k}{n}\right),\ k=0,1,2,\dots,n\right\}.$$

Using this representation, part c) of the lemma and the fact that

$$\mathbf{P}\left\{\sup_k|\zeta_{nk}|>\log n\right\}\leqslant\frac{1}{n^2},$$

one can establish (1).

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References

1. A. Ya. Khinchin, *Asymptotic Laws of Probability Theory*, Moscow, 1936.

2. Yu. V. Prokhorov, Theory Probab. Appl., **1**, 177 (1956).

Note: Figure translations are in progress. See original paper for figures.