



Soviet-era science, translated into English

Physics

1960

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Abstract

Full Text

Physics

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A GENERALIZATION OF GOR' KOV' S METHOD TO THE CASE OF ELECTRON-PHONON INTERACTION

(Presented by Academician A. F. Ioffe, 23 I 1960)

The theory of superconductivity by the Green' s-function method was first constructed by Gor' kov on the basis of a model four-fermion interaction (1). In the present work the theory of superconductivity is constructed by the same method on the basis of the real interaction of electrons with phonons (without taking the Coulomb interaction into account). Using the results of (2), we do not make the assumption that the coupling constant is small. The case $T = 0$ is considered.

The equations of motion for the Heisenberg operators of the electron field ψ, ψ^+ and the field of longitudinal phonons $\vec{\varphi}$ have the form

$$\begin{aligned} \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) \psi(x) &= -g\psi(x) \operatorname{div} \vec{\varphi}(x), \\ \left(\frac{\partial^2}{\partial t^2} - \nabla^2 \right) \vec{\varphi}(x) &= -g \operatorname{grad}[\psi^+(x)\psi(x)]. \end{aligned} \quad (1)$$

Notation: g is the coupling constant; $\hbar = s = 1$; s is the speed of sound; $x = \{\mathbf{x}, t\}$.

The solution of the second equation can be written in the form (3)

$$\vec{\varphi}(x) = \vec{\varphi}_0(x) - g \int d^4x' D_F^0(x-x') \operatorname{grad}[\psi^+(x')\psi(x')]; \quad (2)$$

D_F^0 is the causal Green' s function of free phonons. As a particular solution the half-sum of the retarded and advanced solutions has been chosen. From (1), (2) it is not difficult to obtain equations for the functions

$$\hat{G}(x-x') = -i\langle T\psi(x)\psi^+(x') \rangle,$$

$$\hat{F}(x-x') = \exp\{2i\mu t\} \langle N | T \psi(x) \psi(x') | N + 2 \rangle;$$

$$\hat{F}^+(x-x') = \exp\{-2i\mu t\} \langle N + 2 | T \psi^+(x) \psi^+(x') | N \rangle$$

($|N\rangle$ is the ground state of the system with N electrons; μ is the chemical potential of the electrons), separating the averages over the ground state of the T -product of four ψ -operators in the same way as was done in (1). As a result we obtain the system of equations:

$$\begin{aligned} \left(i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} \right) \hat{G}(x-x') &= \delta^4(x-x') + ig^2 \int d^4x'' D(x-x'') \left[\hat{G}(x-x'') \hat{G}(x''-x') - \right. \\ &\quad \left. - \hat{F}^+(x''-x') \hat{F}(x-x'') \exp\{-2i\mu(t-t'')\} \right], \end{aligned} \quad (3)$$

$$\begin{aligned} \left(-i \frac{\partial}{\partial t} + \frac{\nabla^2}{2m} + 2\mu \right) \hat{F}^+(x-x') &= ig^2 \int d^4x'' D(x-x'') \left[\hat{F}^+(x-x'') \hat{G}(x''-x') + \right. \\ &\quad \left. + \hat{G}(x''-x') \hat{F}^+(x''-x) \exp\{-2i\mu(t-t'')\} \right]. \end{aligned}$$

In deriving (3) we replaced the free Green function of the phonon by the exact one and omitted the terms arising from this replacement and from $\tilde{\varphi}_0$. Below it will be shown that, for a nonsuperconducting metal, such a neglect corresponds to replacing, in the Dyson equation for G , the vertex part by unity, which means expanding it in powers of s/v_F . In addition, we have denoted $\nabla^2 D_F = D$. It can be shown that in the given model of a superconductor $F = F^+$, and therefore we do not write down the equation for F . The dependence on spinor indices may be chosen in the form

$$\hat{G}_{\alpha\beta} = G \cdot \delta_{\alpha\beta}; \quad \hat{F}_{\alpha\beta}^+ = F^+ \cdot I_{\alpha\beta}; \quad \hat{I} = i \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}; \quad \hat{I}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (4)$$

Using (4) and passing in (3) to the Fourier representation, we obtain

$$\begin{aligned} [\omega - \xi_k - \Sigma(k, \omega)] G(k, \omega) &= 1 - if(k, \omega) \cdot F^+(k, \omega); \\ [\omega + \xi_k + \Sigma(k, -\omega)] F^+(k, \omega) &= -if(k, \omega) \cdot G(k, \omega). \end{aligned} \quad (5)$$

Fig. 1

Figure 1: Fig. 1

Notation:

$$\omega \rightarrow \omega - \mu; \quad \xi_k = k^2/2m - \mu; \quad \Sigma = \hat{D}G, \quad f = \hat{D}F^+;$$

$$\hat{D}\varphi \equiv g^2(2\pi)^{-4} \int d^3k' d\omega' D(k - k'; \omega - \omega')\varphi(k', \omega');$$

$D(k, \omega)$ is the Fourier component of the function $D(x)$; it may be taken from (2).

From (5) we obtain

$$G(k, \omega) = \frac{\omega + \xi_k + \Sigma(k, -\omega)}{\Pi(k, \omega)}; \quad F^+(k, \omega) = -i \frac{f(k, \omega)}{\Pi(k, \omega)};$$

$$\Pi(k, \omega) = [\omega - \xi_k - \Sigma(k, \omega)][\omega + \xi_k + \Sigma(k, -\omega)] - f^2(k, \omega). \quad (6)$$

If in (6) we pass to a nonsuperconducting metal ($F^+ = 0$), then we obtain the Dyson equation for G with $\Gamma = 1$. Since there is no reason to believe that the interaction of electrons with phonons in a nonsuperconducting metal differs essentially from that in a superconductor, one may hope that the system (6), describing a superconductor, is written with accuracy up to terms of order s/v_F . If in (5) D is replaced by unity, then (5) goes over into the system of equations obtained by Gor'kov, i.e., Gor'kov's equations are obtained from (5) by neglecting the mean energy of electronic transitions in comparison with the mean energy of phonons, which is the usual condition under which the four-fermion Hamiltonian is obtained from the Fröhlich Hamiltonian.

Fig. 1

From (6), acting with the operator \hat{D} , we obtain a system of equations for f and Σ :

$$f = -i\hat{D}\frac{f}{\Pi}; \quad \Sigma = i\hat{D}G. \quad (7)$$

To solve (7) it is necessary to know the properties of the analytic continuations Σ and f into the domain of complex ω . It is not difficult to write spectral expansions of the Chew-Lehmann type for G, F, F^+ , from which the following properties follow: the spectral expansions define functions of the complex variable $\omega, \tilde{G}, \tilde{F}, \tilde{F}^+$, analytic in the whole plane except for the contour L (Fig. 1).

The values of \tilde{G} on the banks of L are complex conjugates. The values of \tilde{F} and \tilde{F}^+ on the banks of L possess

the following property: $\tilde{F}^*(\omega + i0) = \tilde{F}^+(\omega - i0)$, but since $F = F^+$, the values of \tilde{F}^+ are also complex conjugate on the banks of L . Hence it follows that one can introduce functions of the complex variable ω , $\tilde{\Sigma}$ and \tilde{f} , which coincide on the real axis with Σ and f , are analytic in the whole plane with the exception of the contour L , and on the banks of L the values of each of these functions are complex conjugate.

In the first equation (7) let us pass to integration over $q = |\mathbf{k} - \mathbf{k}'|$, $\xi' = k'^2/2m - \mu$, and divide the region of integration over ξ' into two parts: 1) $-\alpha < \xi' < \alpha$, 2) $|\xi'| > \alpha$, where $\omega_0 \ll \alpha \ll \mu$ (ω_0 is the mean phonon frequency). In the integral over region 2), in the integrand one may neglect f, Σ, ω' in comparison with ξ' ; therefore the integral over region 2) is a constant for $k \sim k_F$. Further, introducing the function

$$\sigma(\omega) = \Sigma(k, \omega) + \frac{i\rho}{4\pi} \int_0^{q_1} q dq \int_{-\infty}^{+\infty} d\omega' D(q, \omega') \int_{\xi_{k-q}}^{\xi_{k+q}} P \frac{d\xi'}{\xi'},$$

where P denotes the principal value of the integral, and including in the integrand of (7) the above-mentioned constant in μ , we obtain the following equation for σ :

$$\sigma(\omega) = -i \frac{\rho}{4\pi} \int_0^{q_1} \frac{q^3 dq}{\omega_q} \int_{-\infty}^{+\infty} d\omega' \left[\frac{1}{\omega_q - \omega + \omega' - i\delta_q} + \frac{1}{\omega_q + \omega - \omega' - i\delta_q} \right] \int_{-\infty}^{+\infty} d\xi' \frac{\omega' - \sigma(\omega')}{\Pi(k', \omega')}, \quad (8)$$

where we have substituted the explicit expression for $D(k, \omega)$ and, because of the rapid convergence of the integral over ξ' , have replaced the integration limits $\pm\alpha$ by $\pm\infty$; $\rho = g^2 m / 2\pi^2 k_F$; $\Pi(\xi, \omega) = [\omega - \sigma(\omega)]^2 - \xi^2 - f^2(\omega) \equiv -[\xi^2 - \chi^2(\omega)]$; $\sigma(\omega) = -\sigma(-\omega)$; $q_1 = \min\{q_m, 2k_F\}$; q_m is the Debye momentum.

The integral over ξ' is evaluated elementarily. We transform the integral over ω' as follows: we pass to the analytic functions $\tilde{\sigma}$ and $\tilde{\chi}$; then, in the plane of the complex variable ω' , one can deform the path of integration into L_1 in the first term of the D -function and into L_2 in the second term. Then by the change $\omega' \rightarrow -\omega'$ we transform $L_2 \rightarrow L_1$ and, taking into account that $\tilde{\sigma}$ and $\tilde{\chi}$ are complex conjugate on the banks of L_1 and $\text{Im } \chi > 0$ for $\omega > 0$, we finally obtain

$$\sigma(\omega) = -\frac{\rho}{2} \int_0^{q_1} \frac{q^3 dq}{\omega_q} \int_{\Delta}^{\infty} d\omega' \text{Re} \left[\frac{\omega' - \sigma(\omega')}{\chi(\omega')} \right] \left[\frac{1}{\omega_q - \omega + \omega' - i\delta_q} - \frac{1}{\omega_q + \omega + \omega' - i\delta_q} \right]. \quad (9)$$

Carrying out the analogous procedure, we obtain for $f(\omega)$

$$f(\omega) = \frac{\rho}{2} \int_0^{q_1} \frac{q^3 dq}{\omega_q} \int_{\Delta}^{\infty} d\omega' \operatorname{Re} \left[\frac{f(\omega')}{\chi(\omega')} \right] \left[\frac{1}{\omega_q - \omega + \omega' - i\delta_q} + \frac{1}{\omega_q + \omega + \omega' - i\delta_q} \right]. \quad (10)$$

Δ , which appears in these formulas, is formally defined as the smallest energy of an elementary excitation. Let us find its relation to the desired functions. The spectrum of elementary excitations is determined by the roots of the equation $\Pi(\xi, \omega) = 0$. Substituting $\omega = \varepsilon + i\gamma$, $f = f_1 + if_2$, $\sigma = \sigma_1 + i\sigma_2$, and assuming $\varepsilon > \gamma$, we find, for $\xi \ll \omega_0$,

$$\varepsilon = \pm \sqrt{\tilde{\xi}^2 + \Delta^2}; \quad \tilde{\xi} = \frac{\xi}{1 - \sigma'_1(0)}; \quad \Delta = \frac{f_1(0)}{1 - \sigma'_1(0)}. \quad (11)$$

It has been taken into account that $f_2(0) = \sigma_2(0) = 0$. For the damping γ , for arbitrary ξ we have

$$\gamma(\xi) = \frac{[\xi - \sigma_1(\xi)]\sigma_2(\xi) + f_1(\xi)f_2(\xi)}{[1 - \sigma'_1(\xi)][\xi - \sigma_1(\xi)]}; \quad \gamma(\xi) = -\gamma(-\xi). \quad (12)$$

The real part of (10) at $\omega = 0$ determines Δ . From (9), (10) one can obtain the following information concerning f_2 and σ_2 : $f_2(\omega) = \sigma_2(\omega) = 0$ for $|\omega| \ll \Delta$, i.e., the damping γ at $\xi = 0$ vanishes; for $(\omega - \Delta)/\Delta \ll 1$, for f_2 and σ_2 we have the expressions

$$\frac{f_2(\omega)}{f_1(0)} = -\frac{\sigma_2(\omega)}{\Delta} = -\frac{4\pi}{15\sqrt{2}} \rho \left(\frac{\Delta}{a} \right)^2 \left(\frac{\omega - \Delta}{\Delta} \right)^{5/2},$$

$$a = \frac{g^2 m}{8\pi^2 k_F} \left(1 - \frac{g^2 k_F m}{\pi^2} \right)^{-1}.$$

For $\Delta = 0$, (9) goes over into the corresponding formula for a nonsuperconducting metal when $\omega/\omega_0 \gg \omega_0/\mu$ ⁽²⁾.

In conclusion I express my gratitude to L. E. Gurevich for numerous discussions and advice.

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Received
17 XII 1959

CITED LITERATURE

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