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Abstract

Full Text

Mathematics

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On Locally Bicompact Solvable Groups Satisfying the Minimality Condition for Closed Subgroups

(Presented by Academician A. I. Mal' cev on 15 III 1959)

As in the case of abstract groups, a topological group G is called locally nilpotent (locally solvable) if every finite set of its elements generates a nilpotent (solvable) subgroup. V. M. Glushkov ([4]) studied locally bicompact locally nilpotent, as well as locally solvable, groups satisfying the following condition (the minimality condition for closed subgroups):

(L) Every decreasing chain of closed subgroups

$$A_1 \supset A_2 \supset \cdots$$

terminates after a finite number of terms.

On the other hand, still earlier S. N. Chernikov ([3]) investigated the structure of abstract locally nilpotent and locally solvable groups whose abelian subgroups satisfy the condition that descending chains of subgroups terminate. The natural analogue of this condition is the following property of topological groups:

(M) Every closed abelian subgroup A of the group G satisfies the minimality condition for closed subgroups.

The properties (L) and (M) are equivalent for certain classes of groups. The following theorem holds:

Theorem. *A locally bicompact solvable group satisfies the minimality condition for closed subgroups if and only if every closed abelian subgroup of it satisfies the minimality condition for closed subgroups.*

The proof of this theorem is carried out with the aid of a lemma:

Lemma. *Let the connected component of the identity K of a locally bicompact group G be a finite-dimensional toroidal group. The factor group G/K will be finite if the group G satisfies condition (M) and, in addition, at least one of the following two conditions holds:*

a) *the factor group G/K is abelian and bicompact;*

b) *the factor group G/K is discrete and decomposes into a direct product of cyclic subgroups of prime orders.*

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We outline the proof of the theorem. Obviously, it suffices to prove that if a locally bicompact solvable group G satisfies condition (M), then it also satisfies condition (L).

Let G satisfy condition (M). Then all its elements are bicompact. Further, if K is the connected component of the identity of the group G, then, by Theorem 1 of the paper of A. I. Mal' cev ([1]), in K there exists a bicompact central subgroup Z such that the factor group K/Z is a Lie group. The group K/Z is compact, since otherwise it would contain a one-dimensional vector subgroup (see ([2])), which is impossible in view of the bicompactness of all elements of the group K. Hence the subgroup K is bicompact. Being connected, solvable, and bicompact, it is abelian. Since, moreover, it satisfies the minimality condition for closed subgroups, it follows from ([4]) that it is a finite-dimensional toroidal group.

Let $\overline{H}$ be some closed abelian subgroup of the factor group G/K; let H be its full preimage in G. All elements of the locally bicompact group $\overline{H}$ are bicompact, and therefore it has a bicompact subgroup $\overline{B}$ such that the factor group $\overline{H}/\overline{B}$ is discrete. If B is the full preimage of $\overline{B}$ in G, then condition a) of the lemma is fulfilled for it. Hence $\overline{B}$ is finite, and therefore discrete. Consequently, $\overline{H} = H/K$ is a discrete periodic group.

Denote by P_1 and $P_2, \ldots$ all the Sylow subgroups of the group $\overline{H}$, corresponding to certain prime numbers $q_1, q_2, \ldots$, and by $Q_1, Q_2, \ldots$ their lower layers, i.e. the totalities of all elements of prime orders.

If Q_i is the full preimage of $\overline{Q}_i$ $(i = 1, 2, \ldots)$ in G, then Q_i satisfies condition b) of the lemma. Hence $\overline{Q}_i$ is a finite group for all indices $i = 1, 2, \ldots$. The set $P_1, P_2, \ldots$ of all Sylow subgroups distinct from 1 cannot be infinite. Indeed, if from each layer $\overline{Q}_1, \overline{Q}_2, \ldots$ one chooses one element not equal to 1, then all these elements form a group $\overline{Q}$, decomposable into the direct product of cyclic subgroups of prime orders. The full preimage Q of this group $\overline{Q}$ in G satisfies condition b) of the lemma. Hence $\overline{Q}$ is finite and, consequently, the set of all Sylow subgroups is finite.

Thus we have proved that the discrete group $\overline{H}$ is decomposable into the direct product of a finite number of Sylow subgroups, whose lower layers are finite. But this means that $\overline{H}$ satisfies the minimality condition for subgroups. Consequently, every closed commutative subgroup of the factor group G/K is discrete and satisfies the minimality condition for subgroups.

Consider the factor group $\overline{G} = G/K$ as an abstract group and, relying on the results of S. N. Chernikov (3), we obtain that $\overline{G}$ satisfies the minimality condition for subgroups. Therefore $\overline{G}$ has a subgroup of finite index decomposable into the direct product of a finite number of subgroups of type q^{∞} for certain prime numbers q. From this it is easy to conclude that $\overline{G}$ is a discrete group and satisfies condition (L). Since the subgroup K is compact and satisfies condition (L), the group G also satisfies this condition.

Corollary. If every closed abelian subgroup of a locally bicompact solvable

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group G satisfies the minimality condition for closed subgroups, then G has a closed subgroup of finite index which decomposes into the direct product of a finite number of one-dimensional toroidal subgroups and discrete subgroups of type q^∞ for certain prime numbers q.

The proof is obtained with the aid of Theorem 2 and the results of V. M. Glushkov (4).

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Note: Figure translations are in progress. See original paper for figures.

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