Reports of the Academy of Sciences of the USSR

1960

SovietRxiv

Abstract

Full Text

Reports of the Academy of Sciences of the USSR

1960. Volume 133, No. 6

MATHEMATICS

Z. V. REITBLAT

A MEAN-VALUE THEOREM FOR LINEAR ELLIPTIC EQUATIONS WITH COEFFICIENTS OF THE LIPSCHITZ CLASS

(Presented by Academician S. L. Sobolev on 15 IV 1960)

In the present paper a mean-value theorem is proved for solutions of the equation

$$\mathfrak{M}u = \sum_{i,k}^{m} a_{ik}u_{ik} + \sum_{i} b_i u_i + cu = f \tag{1}$$

in an m-dimensional open domain $\Omega = \Omega(x)$ with boundary S, under the assumption that* $f \in L_p$; $p > m/2$; $a_{ik}, b_i, c \in C^{(0,\lambda)}$, $0 < \lambda \le 1$; $c < 0$; $\sum_{i,k}^{m} a_{ik}t_i t_k > \gamma \sum_i t_i^2$, $\gamma > 0$.

Namely, it is proved that the solution of equation (1), for arbitrary $y \in \Omega$, $|y - S| > \delta$, satisfies the equality

$$u(y) = \frac{1}{\varkappa(\delta)} X(y;\delta) + \frac{1}{\varkappa(\delta)} \int_{\Omega} F^{(\delta)}(y,z)u(z)\, d\Omega \tag{2}$$

(the definition of the functions $\varkappa(\delta), X(y;\delta), F^{(\delta)}(y,z)$ will be given below), and conversely, every function $u \in L_p$ for which equality (2) is valid is a solution of equation (1).

This was proved by other authors under the assumption that $a_{ik} \in C^{(2,\lambda)}, \lambda \ge 0$, and the concept of the adjoint operator was used (see the bibliography in (1)). In the present paper, the proof uses the maximum principle with the Green' s function for equation (1) for a sphere of small radius.

Let $y \in \Omega$, and let δ be so small that the sphere $\Omega_\delta(y)\{|x-y| < \delta\}$ lies entirely in Ω together with its boundary $S_\delta(y)$. Let $F(x,z;y,\delta)$, $x, z \in \Omega_\delta(y)$, be the Green's function for solving the Dirichlet problem for equation (1) in the domain

$\Omega_\delta(y)$, so that the solution of this problem in the domain $\Omega_\delta(y)$ is given by the formula ($u \in C(S_\delta)$, $f \in L_p(\Omega_\delta)$)

$$u(x) = -\int_{\Omega_\delta(y)} F(x,z;y,\delta) f(z)\, d\Omega_z - \int_{S_\delta(y)} \mathfrak{D}_z F(x,z;y,\delta) u(z)\, dS_z. \quad (3)$$

Put $\delta = \rho$, $x = y$ in (3), multiply both sides of the resulting equality by $\varphi_\delta(\rho)\rho^{m-1}$, where

$$\varphi_\delta(\rho) = \exp\left(-\frac{1}{\delta^2 - \rho^2}\right)$$

for $\rho < \delta$ and $\varphi_\delta(\rho) = 0$ for $\rho \geq \delta$, and integrate with respect to ρ from 0 to δ. We obtain

$$\varkappa(\delta) u(y) = X(y;\delta) + \int_{\Omega_\delta(y)} F^{(\delta)}(y,z) u(z)\, d\Omega_z, \quad (2_1)$$

* The notation of the monograph (1) is used.

where

$$\varkappa(\delta) = \int_0^\delta \varphi_\delta(\rho)\rho^{m-1}\, d\rho; \quad (2_2)$$

$$X(y;\delta) = -\int_0^\delta \varphi_\delta(\rho)\rho^{m-1} \int_{\Omega_\rho(y)} F(y,z;y,\rho) f(z)\, d\Omega_z\, d\rho; \quad (2_3)$$

$$F^{(\delta)}(y,z) = -\varphi_\delta(\rho)\rho^{m-1}\mathfrak{D}_{zF}(y,z;y,\rho). \quad (2_4)$$

Here in formula (2_4) $\rho = \rho(z) = |z-y|$, and $\mathfrak{D}_z$ means that the operator $\mathfrak{D}$ is applied to F as a function of z on the surface of the sphere $|z-y| = \rho$. The integral over $\Omega_\delta(y)$ in formula (2_1) may be regarded as extended over Ω, since $F^{(\delta)} \equiv 0$ for $z \notin \Omega_\delta(y)$.

Lemma 1. If $u \in L_p(\Omega)$, $\psi(y) = \int_\Omega F^{(\delta)}(y,z) u(z)\, d\Omega_z$, then $\psi(y) \in C(\widetilde{\Omega}_\delta)$, where $\widetilde{\Omega}_\delta \subset \Omega$, $|\widetilde{\Omega}_\delta - S| > \delta$.

Lemma 2. If $f \in L_p(\Omega)$, then $X(y;\delta) \in C(\widetilde{\Omega}_\delta)$.

The proof of Lemmas 1 and 2 is based on the properties of integral equations for the construction of the Green function ([1−3]).

Lemma 3. If $u \in C(\Omega)$ and, for all y and all sufficiently small δ such that $\Omega_\delta(y) \subset \Omega$, the equality

$$u(y) = \frac{1}{\varkappa(\delta)} \int_{\Omega_\delta(y)} F^{(\delta)}(y,z) u(z)\, d\Omega_z, \tag{4}$$

holds, then $u(x)$ has in Ω neither a positive maximum nor a negative minimum.

Proof. 1. We have $-\mathfrak{D}_{zF}(y,z;y,\rho) > 0$, for otherwise $u(x)$—the solution of the Dirichlet problem for equation (1) with $f = 0$, equal to zero on $S_\rho(y)$ everywhere except for a small neighborhood of a point $z \in S_\rho(y)$, in which $u > 0$—would be negative at the center of the sphere $\Omega_\rho(y)$, which is impossible.

2. Next, if $v(x)$ is the solution of the Dirichlet problem for equation (1) with $f = 0$, $v|_{S_\rho(y)} = 1$, then $0 < v(y) < 1$, and therefore, according to (3),

$$0 < -\int_{S_\rho(y)} \mathfrak{D}_{zF}(y,z;y,\rho)\, dS_z < 1.$$

Multiplying this inequality by $\varphi_\rho(\rho)\rho^{m-1}$ and integrating with respect to ρ from 0 to δ, we obtain

$$0 < \frac{1}{\varkappa(\delta)} \int_{\Omega_\delta(y)} F^{(\delta)}(y,z)\, d\Omega_z < 1. \tag{5}$$

3. Let $u(x)$ satisfy the conditions of the lemma. Suppose that $u(x)$ attains a maximum at $x = y$, $u(y) > 0$. Then, for sufficiently small δ, one has $u(z) < u(y)$, $z \in \Omega_\delta(y)$, $z \neq y$, and, by virtue of inequality (5),

$$\frac{1}{\varkappa(\delta)} \int_{\Omega_\delta(y)} F^{(\delta)}(y,z) u(z)\, d\Omega_z < \frac{1}{\varkappa(\delta)} \int_{\Omega_\delta(y)} F^{(\delta)}(y,z) u(y)\, d\Omega_z < u(y),$$

which contradicts identity (4). The absence of negative minima of the function $u(x)$ in Ω is proved analogously.

Lemmas 1-3 make it possible to prove the following theorem.

Theorem (mean-value theorem). If $u(x)$ is a solution of equation (1), then for all $y \in \Omega$ and all $\delta < |y - S|$ equality (2) is satisfied. Convers-

but if $u \in L_p$ and for all $y \in \Omega$, for all $\delta < \delta_1/2$, where $\delta_1 < |y - S|$ is arbitrary, equality (2) holds, then u is a solution of equation (1).

Proof. The first part of the theorem is obvious; let us prove the second. Thus, let $u(x)$ satisfy the conditions of the theorem. Let $\widetilde{\Omega}_{\delta_1} \subset \Omega$ be such a subdomain of Ω that $|\widetilde{\Omega}_{\delta_1} - S| > \delta_1$. Then, by Lemmas 1 and 2, we obtain $u \in C(\widetilde{\Omega}_{\delta_1})$. We may assume that $\widetilde{S}_{\delta_1}$, the boundary of $\widetilde{\Omega}_{\delta_1}$, belongs to the class $A^{(2,\lambda)}$. Construct $\bar{u}$, a solution of equation (1), such that $(\bar{u} - u)|_{\widetilde{S}_{\delta_1}} = 0$. Then $\bar{u}$ satisfies equality (2), and the difference $\bar{u} - u$ satisfies equality (4). But then, by Lemma 3, from the fact that $(\bar{u}-u)|_{\widetilde{S}_{\delta_1}} = 0$ it follows that $\bar{u}-u = 0$ in Ω. Since δ_1 is arbitrary, $u(x)$ is a solution of equation (1) in the whole domain Ω, as was required to prove.

Corollary 1. The totality of solutions of equation (1) for $f = 0$ forms a subspace in L_p.

Corollary 2. If a sequence $u_n \in L_p \cap D_{\mathfrak{M}}$ is such that $\|u_n - u\|_{L_p} \to 0$, $\|\mathfrak{M}u_n - f\|_{L_p} \xrightarrow[n\to\infty]{} 0$, then u is a solution of equation (1).

The validity of the corollaries follows from the fact that, in equality (2), written for u_n, passage to the limit as $n \to \infty$ under the integral sign is permissible.

In conclusion I express my gratitude to Yu. G. Reshetnyak for advice in writing the present article.

Institute of Mathematics Siberian Branch of the Academy of Sciences of the USSR

Received 6 IV 1960

REFERENCES

1. K. Miranda, *Equations with Partial Derivatives of Elliptic Type*, IL, 1957.

2. G. Giraud, C. R., 202, 380 (1936).

3. G. Giraud, Bull. Sci. Math., 61, 172 (1937).

Note: Figure translations are in progress. See original paper for figures.