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Fig. 1

Figure 1: Fig. 1

Abstract

Full Text

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"UNPHYSICAL" THRESHOLDS IN PERTURBATION THEORY

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For most reactions between elementary particles, the energy spectrum of the absorptive part of the matrix element is determined by the energy of the intermediate state. However, in a number of cases a discontinuous spectrum appears that cannot be explained by simple physical considerations; thus there arise "unphysical," anomalous thresholds, first discovered by Karplus, Sommerfield, and Wichmann ([1,2]).

Fig. 1

The study of such thresholds is of all the greater interest because their presence or absence directly affects the possibility of writing Mandelstam spectral representations, as follows from the works of Mandelstam himself ([3]), Tarski ([4]), Vladimirov ([5]), and others. However, up to now in this respect only the lowest orders of perturbation theory have been studied by a number of authors, and only the investigation of Nakanishi ([6]) contains a general approach to the problem of finding anomalous spectra.

The main result of the present work is the formulation of a sufficient condition for the presence of an anomalous threshold for a diagram of arbitrary complexity. Let us consider the diagram of Fig. 1, in which the lines $1, 2, \ldots, r$ are singled out, forming an intermediate state with the least mass $\sum_{i=1}^{r} m_i$ between the states $p_1, p_2, \ldots, p_L$ and $p_1', p_2', \ldots$. In general, the matrix element corresponding to this diagram is a function of the variable $P^2 = \left(\sum_{i=1}^{L} p_i\right)^2$, analytic on the real axis from $-\infty$ to some value P_0^2.

If the threshold value P_0^2 is equal to the square of the sum of the masses $\left(\sum_{i=1}^{r} m_i\right)^2$ on the cut that severs all the lines of the intermediate state (in Fig. 1 it is shown by a dashed line), the threshold is "physical" ; if, however, the singularities begin at smaller values of P^2, we shall speak of an "unphysical" threshold.

According to the widely known α-representation, the matrix element corresponding to the diagram of Fig. 1 has the form:

$$G(P^2) \sim \int \Pi \, d\alpha \, \frac{h(\alpha)\, \delta(1-\Sigma\alpha)}{[Q(P^2)]^c}. \tag{1}$$

Here, in agreement with the work of Symanzik ([7]), it is put that

$$Q(P^2, \alpha) = I(P^2, \alpha) - M(\alpha) D(\alpha).$$

To each i-th line of the diagram there are assigned a parameter α_i and a mass m_i. The form $M(\alpha)$ is the sum $\sum_i \frac{m_i^2}{\alpha_i}$ over all lines of the diagram; $h(\alpha)$ is an integrable function of the variables α; the function $D(\alpha)$ is constructed from α according to the following rules. We shall call a skeleton of a diagram a set of its lines which forms a maximally weakly connected diagram (one that is broken by removal of any line) and passes through all vertices of the original graph. In order to obtain D, one must form the product of the parameters α belonging to the lines of some skeleton, and sum such products over all possible skeletons of the diagram. Thus,

$$D(\alpha) = \sum_{\text{skel}} \prod \alpha.$$

I is a quadratic form of the external momenta p of the form:

$$I = \sum_{\text{div}} D'(\alpha') D''(\alpha'') \left(\sum p''\right)^2.$$

Each term of this sum is obtained as follows. We draw a cut dividing the diagram into two parts, form the functions D' and D'', respectively, for the first and second parts, and multiply their product by the square of the sum of the external momenta entering, for example, the second part. Summation of such expressions over all possible divisions of the diagram gives the quadratic form I.

It is now clear that a necessary condition for singularity of G at the point $P^2 = P_0^2$ is the fulfillment of the relation $Q(P_0, \alpha) = 0$, or of the equivalent equality $I(P_0, \alpha)/MD = 1$, for some α in the region of integration. As Tarski showed, the point P_0^2 will be a point of lowest singularity only when, for $P^2 < P_0^2$, the equality $I(P, \alpha)/MD = 1$ is not fulfilled anywhere in the region of integration.

Let us now introduce, for the lines of the intermediate state $1, 2, \dots, r$, new variables λ and x: $\alpha_1 = \lambda x_1, \; \alpha_2 = \lambda x_2, \dots, \alpha_r = \lambda x_r$, with the additional condition

$$\sum_{i=1}^{r} x_i = 1,$$

and for the remaining lines the variables y: $\alpha_{r+1} = (1-\lambda)y_{r+1}, \dots;$

$$\sum y = 1.$$

Theorem 1. *The equality $I/MD = 1$ is fulfilled for*

$$P^2 = P_0^2 = \left(\sum_{i=1}^{r} m_i\right)^2$$

when $\lambda = 0$, $x_i^0 = m_i \Big/ \sum_{j=1}^{r} m_j$, and, consequently, the point $P^2 = P_0^2$ is, generally speaking, singular.

Proof. The quadratic form

$$\sum_{\text{div}} D'D'' \left(\sum p''\right)^2$$

is a polynomial in the parameter λ. Recall that, for example,

$$D' = \sum_{\text{skel}} \prod \alpha'.$$

Each skeleton of the diagram D' passes through all its vertices, and parts 1 and 2 of the diagram in Fig. 1 are connected only by the lines $1, 2, \dots, r$. All cuts in the diagram of Fig. 1 that do not coincide with those marked by the dotted line divide it into such parts that in at least one of them there is a group of vertices connected with the other vertices only by lines from among $1, 2, \dots, r$. Therefore the terms in the form I corresponding to these cuts contain λ to a higher power than zero. As λ tends to zero, only one term thus remains in the form I,

$$I \longrightarrow D_1 D_2 \left(\sum_{j=1}^{L} p_j\right)^2 = D_1 D_2 P^2,$$

where D_1 and D_2 correspond to parts 1 and 2 of the diagram in Fig. 1.

It is also clear that the terms of minimal first degree in λ in the function D have the form:

$$D = D_1 D_2 \lambda \left(\sum x\right)' + O(\lambda^2) = \lambda D_1 D_2.$$

Consequently, the only term in the expression MD that does not vanish as λ tends to zero is the term

Fig. 2

Figure 2: Fig. 2

$$\frac{1}{\lambda}\left(\sum_{i=1}^{r}\frac{m_i^2}{x_i}\right)\lambda D_1 D_2 = \left(\sum_{i=1}^{r}\frac{m_i^2}{x_i}\right) D_1 D_2.$$

It is now clear that the ratio I/MD tends in the limit to the expression

$$P^2 \Bigg/ \sum_{i=1}^{r}\frac{m_i^2}{x_i}.$$

The maximum of this expression is realized for

$$x_i^0 = m_i \Bigg/ \sum_{i=1}^{r} m_i$$

and is equal to

$$P^2 \Bigg/ \left(\sum_{i=1}^{r} m_i\right)^2.$$

The equality $I(P,\alpha)/MD = 1$ is therefore satisfied for

$$P^2 = P_0^2 = \left(\sum_{i=1}^{r} m_i\right)^2.$$

The theorem is proved.

Fig. 2

The value $P_0 = \sum m$ determines the normal physical threshold only in the case where, in a neighborhood of the point $\lambda = 0$, there is no point at which the ratio $I(P_0,\alpha)/MD$ exceeds unity. Indeed, if such a point existed, then, by virtue of the monotonicity of the function I/MD with respect to P^2 at small λ, there would be a value $P^2 < P_0^2$ for which the ratio I/MD is equal to unity. We thus arrive at the theorem:

Theorem 2. A sufficient condition for the existence of an anomalous threshold for the diagram G is the fulfillment of the relation

$$\frac{\partial}{\partial\lambda}\left[I(P_0,x^0,y) - M(x^0,y)D(x^0,y)\right]\Big|_{\lambda=0} > 0 \qquad (2)$$

for at least some y.

Relation (2) is still inconvenient for concrete application to the study of diagrams. We shall therefore formulate theorem 3.

Theorem 3. A sufficient condition for the presence of an anomalous threshold for the diagram of Fig. 1 is the fulfillment of at least one of the relations:

$$\frac{\partial}{\partial\lambda}\left[I^{(a)}(P_0) - M^{(a)}D^{(a)}\right]\Big|_{\lambda=0} = \frac{\partial}{\partial\lambda}Q^{(a)}\Big|_{\lambda=0} > 0, \quad (3)$$

$$\frac{\partial}{\partial\lambda}\left[I^{(b)}(P_0) - M^{(b)}D^{(b)}\right]\Big|_{\lambda=0} = \frac{\partial}{\partial\lambda}Q^{(b)}\Big|_{\lambda=0} > 0.$$

The diagrams a and b are shown in Fig. 2. Roughly speaking, the content of theorem 3 reduces to the following: the diagram of Fig. 1 has an anomalous threshold if at least one of the diagrams $2a$ or $2b$ has one.

The theorem is easily proved if one takes into account the fact that, up to terms of first order of smallness in λ, the form $Q = I - MD$ for the diagram of Fig. 1 coincides with the expression

$$Q = D_1 Q^{(b)}(P_0) + D_2 Q^{(a)}(P_0).$$

Further, using the Källén–Lehmann spectral representation for the field Green functions, it is not difficult to prove the following assertion:

Theorem 4. *The threshold value P_0^2 (the beginning of the spectrum) does not change when a self-energy part is inserted into any line of a diagram.*

Let us now consider the diagram in Fig. 3. The derivative $\left.\frac{\partial Q(P_0)}{\partial\lambda}\right|_{\lambda=0}$ for this diagram has the form

$$\prod_{j=1}^{r-1}\alpha_{j,j+1}\left\{\sum_{j=1}^{r-1}\frac{1}{\alpha_{j,j+1}}\left(M_1^2\sum_{k=j+1}^{r}m_k + M_2^2\sum_{l=1}^{j}m_l\right)\frac{1}{\sum_{i=1}^{r}m_i} - \right.$$
$$\left. -\sum_{j=1}^{r-1}\frac{m_{j,j+1}^2}{\alpha_{j,j+1}} - \sum_{i<k}m_i m_k\left(\sum_{\xi=i}^{k-1}\frac{1}{\alpha_{\xi,\xi+1}}\right)\right\}.$$

Letting the derivative $\alpha_{j,j+1}$ tend to zero and using Theorem 2, we obtain that a sufficient condition for the existence of an anomalous threshold for this diagram is the fulfillment of at least one of the $r-1$ relations:

Fig. 3 Fig. 4

Fig. 3 **Fig. 4**

$$\frac{M_1^2 \sum_{k=j+1}^{r} m_k + M_2^2 \sum_{l=1}^{j} m_l}{\sum_{i=1}^{r} m_i} - m_{j,j+1}^2 - \sum_{i=1}^{j} m_i \sum_{k=j+1}^{r} m_k > 0 \qquad (j = 1, 2, \dots, r-1). \tag{4}$$

In the case $r = 2$, the conditions (4) coincide with the conditions obtained in the work of Karplus, Sommerfield, and Wichman [(2)].

The diagram of Fig. 4, by Theorem 3, also has an anomalous threshold in the case of fulfillment of at least one of the conditions (4). Of course, everything said remains valid if self-energy parts are included in all lines of the diagrams in Figs. 3 and 4.

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Note: Figure translations are in progress. See original paper for figures.