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# L. R. Rukhadze

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**Abstract**

**Full Text**

**L. R. Rukhadze**

**On Local Duality Theorems**

*(Presented by Academician P. S. Aleksandrov on December 26, 1959)*

**1. Local groups of a space.**

Let  $S^n$  be an  $n$ -dimensional spherical space;  $A$  an arbitrary subset of it;  $x$  an arbitrary point of the set  $A$ ;  $\{U_\lambda\}$  a decreasing directed set of spherical neighborhoods of the point  $x$ ;  $\{F_a\}$  an increasing directed set of all compact subsets of the set  $A$ . The relative homology groups

$$H^r(F_a, F_a - U_\lambda; C)$$

of compact pairs of sets  $(F_a, F_a - U_\lambda)$  over the compact coefficient group  $C$ , with inclusion homomorphisms

$$p_{ab} : H^r(F_a, F_a - U_\lambda; C) \rightarrow H^r(F_b, F_b - U_\lambda; C),$$

which occur for  $a < b$ , generate a direct spectrum of compact groups

$$\{H^r(F_a, F_a - U_\lambda; C), p_{ab}\},$$

whose limiting group in the sense of Chogoshvili (see <sup>(1, 5)</sup>) we shall denote by

$$H^r(A, A - U_\lambda; C).$$

Here the group  $H^r(F_a, F_a - U_\lambda; C)$  is defined as follows (cf. <sup>(2-7)</sup>). To each finite open covering  $O_\alpha$  of the compactum  $F_a$  there is put in correspondence a complex  $K_\alpha$ , the so-called Vietoris complex, whose vertices are the points of  $F_a$ , and a finite subset of them forms a simplex if this subset is contained in one and the same element of the covering  $O_\alpha$ ; the set of such simplices whose vertices belong to  $F_a - U_\lambda$  forms a subcomplex  $L_\alpha$  of the complex  $K_\alpha$ . Consider the increasing directed system of all finite subcomplexes  $K_{\alpha\tau}$  of the complex  $K_\alpha$  and the relative homology group

$$H^r(K_{\alpha\tau}, L_{\alpha\tau}; C),$$

where  $L_{\alpha\tau} = K_{\alpha\tau} \cap L_\alpha$ . The limiting group of the direct spectrum of compact groups

$$H^r(K_{\alpha\tau}, L_{\alpha\tau}; C)$$

with inclusion homomorphisms, taken in the sense indicated above, is, by definition, the group

$$H^r(K_\alpha, L_\alpha; C).$$

The groups  $H^r(K_\alpha, L_\alpha; C)$  and the homomorphisms generated by inclusions

$$(K_\beta, L_\beta) \subset (K_\alpha, L_\alpha),$$

which occur when  $O_\beta$  is inscribed in  $O_\alpha$ ,  $\alpha < \beta$ , form an inverse spectrum, whose limiting group is precisely the group

$$H^r(F_a, F_a - U_\lambda; C).$$

The neighborhoods  $U_\lambda$  may be either spherical or arbitrary (by virtue of the cofinality of the subsystem of spherical neighborhoods in the system of all neighborhoods), which ensures the invariance of this definition and its suitability for arbitrary Hausdorff compact spaces  $F_a$ .

If  $\lambda < \mu$ , then the inclusion

$$(F_a, F_a - U_\lambda) \subset (F_a, F_a - U_\mu)$$

generates a homomorphism

$$\varphi_{\lambda\mu} : H^r(F_a, F_a - U_\lambda; C) \rightarrow H^r(F_a, F_a - U_\mu; C),$$

which, in turn, generates a homomorphism

$$\Phi_{\lambda\mu} : H^r(A, A - U_\lambda; C) \rightarrow H^r(A, A - U_\mu; C)$$

of the limiting groups of the corresponding spectra.

The groups  $H^r(A, A - U_\lambda; C)$ , with the homomorphisms  $\Phi_{\lambda\mu}$  just defined, form a direct spectrum of compact groups

$$\{H^r(A, A - U_\lambda; C), \Phi_{\lambda\mu}\},$$

whose limiting group, taken in the same sense, is the homology group

$$H^r(A, x; C)$$

of the space  $A$  at the point  $x$  over  $C$ .

Let now  $B^s(U_\lambda - F_a; D)$  denote the Betti group of the open set  $U_\lambda - F_a$  over the discrete coefficient group  $D$ . The groups

$$B^s(U_\lambda - F_a; D)$$

with homomorphisms

$$q_{ba} : B^s(U_\lambda - F_b; D) \rightarrow B^s(U_\lambda - F_a; D),$$

generated for  $a < b$  by the inclusions

$$(U_\lambda - F_b) \subset (U_\lambda - F_a),$$

form an inverse spectrum of discrete groups

$$\{B^s(U_\lambda - F_a; D), q_{ba}\},$$

whose limiting group, taken with the discrete topology, we shall denote by

$$B^s(U_\lambda - A; D).$$

If  $\lambda < \mu$ , then the inclusion  $(U_\mu - F_a) \subset (U_\lambda - F_a)$  induces a homomorphism  $f_{\mu\lambda} : B^s(U_\mu - F_a; D) \rightarrow B^s(U_\lambda - F_a; D)$ , and the latter induces a homomorphism  $F_{\mu\lambda} : B^s(U_\mu - A; D) \rightarrow B^s(U_\lambda - A; D)$ .

The groups  $B^s(U_\lambda - A; D)$ , with the homomorphisms  $F_{\mu\lambda}$ , form an inverse spectrum  $\{B^s(U_\lambda - A; D), F_{\mu\lambda}\}$ , whose limit group with the discrete topology is the group  $B^s(S^n - A, x; D)$ .

**2. The product of the groups  $H^r(A, x; C)$  and  $B^s(S^n - A, x; D)$ .**

Let  $r$  and  $s$  be nonnegative integers such that  $r + s = n - 1$ , let  $z^s$  be an  $s$ -dimensional cycle of the domain  $U_\lambda - F$ , where  $F$  is a compact subset of  $S^n$ , and let  $z^r = \{z_\alpha^r\}$  be an  $r$ -dimensional cycle of the set  $F \bmod F - U_\lambda$ , i.e., a cycle of some element of the group  $\dot{H}^r(F, F - U_\lambda; C)$ .

We define the linking coefficient  $\nu(z^s, z_\alpha^r)$  of the cycles  $z^s$  and  $z_\alpha^r$  as the intersection index  $I(f^{s+1}, z_\alpha^r)$ , where  $f^{s+1}$  is a chain of the set  $U_\lambda$  such that  $\Delta f^{s+1} = z^s$ . It is proved that  $\nu(z^s, z_\alpha^r)$  does not depend on the choice of the chain  $f^{s+1}$ , and also that for any  $\alpha, \beta$  the equality  $\nu(z^s, z_\alpha^r) = \nu(z^s, z_\beta^r)$  holds. After this, the linking coefficient of the cycles  $z^s$  and  $z^r$  may be defined as  $\nu(z^s, z^r) = \nu(z^s, z_\alpha^r)$  for any  $\alpha$ .

Let  $b_\lambda \in B^s(U_\lambda - F; D)$  and  $h_\lambda \in H^r(F, F - U_\lambda; C)$ . We shall call the product of these classes the number, independent of the choice of the cycles  $z^s, z^r$ ,

$$b_\lambda \cdot h_\lambda = \nu(z^s, z^r), \quad \text{where } z^s \in b_\lambda, \quad z^r = \{z_\alpha^r\} \in h_\lambda. \quad (1)$$

**Lemma 1.** *The homomorphisms  $\varphi_{\lambda\mu}$  and  $f_{\mu\lambda}$ ,  $\lambda < \mu$ , are conjugate under the multiplication (1).*

Let  $z^s$  be an arbitrary cycle from the homology class  $b_\mu$ ,  $b_\mu \in B^s(U_\mu - F; D)$ . By the definition of the homomorphism  $f_{\mu\lambda}$ , the class  $f_{\mu\lambda}(b_\mu)$  contains the cycle  $z^s$ . Similarly, if  $z^r$  is an arbitrary cycle from the homology class  $h_\lambda$ ,  $h_\lambda \in H^r(F, F - U_\lambda; C)$ , then  $\varphi_{\lambda\mu}(h_\lambda)$  also contains the cycle  $z^r$ .

By the definition of the product of the groups  $B^s(U_\mu - F; D)$  and  $H^r(F, F - U_\mu; C)$ ,

$$b_\mu \cdot \varphi_{\lambda\mu}(h_\lambda) = \nu(z^s, z^r), \quad (2)$$

and by the definition of the product of the groups  $B^s(U_\lambda - F; D)$  and  $H^r(F, F - U_\lambda; C)$ ,

$$f_{\mu\lambda}(b_\mu) \cdot h_\lambda = \nu(z^s, z^r). \quad (3)$$

(2) and (3) give

$$b_\mu \cdot \varphi_{\lambda\mu}(h_\lambda) = f_{\mu\lambda}(b_\mu) \cdot h_\lambda,$$

which was required to prove.

Let us now consider an arbitrary subset  $A$  of  $S^n$ . Let  $\{F_a\}$  be the directed set of all compact subsets of the set  $A$ ; let  $b_\lambda = \{b_\lambda^a\}$  be an arbitrary element of the group  $B^s(U_\lambda - A; D)$ ; and let  $h_\lambda = \{h_\lambda^a\}$  be an arbitrary element of the group  $H^r(A, A - U_\lambda; C)$ . It is proved that the product  $b_\lambda^a \cdot h_\lambda^a$  introduced above does not depend on the choice of the index  $a$ , which gives grounds for defining the product of the elements  $b_\lambda$  and  $h_\lambda$  by the equality

$$b_\lambda \cdot h_\lambda = b_\lambda^a \cdot h_\lambda^a \quad \text{for any } a. \quad (4)$$

**Lemma 2.** *The homomorphisms  $\Phi_{\lambda\mu}$  and  $F_{\mu\lambda}$ ,  $\lambda < \mu$ , are conjugate under the multiplication (4).*

Indeed, by the definition of the multiplication (4) we have  $b_\mu \cdot \Phi_{\lambda\mu}(h_\lambda) = b_\mu^a \cdot \varphi_{\lambda\mu}(h_\lambda^a) = f_{\lambda\mu}(b_\mu^a) \cdot h_\lambda^a = F_{\mu\lambda}(b_\mu) \cdot h_\lambda$ , which was required to prove.

Finally, let us define the product of elements  $b = \{b_\lambda\}$  and  $h = \{h_\lambda\}$  of the groups  $B^s(S^n - A, x; D)$  and  $H^r(A, x; C)$  by means of the equality

$$b \cdot h = b_\lambda \cdot h_\lambda, \quad (5)$$

where  $b_\lambda \in B^s(U_\lambda - A; D)$  and  $h_\lambda \in H^r(A, A - U_\lambda; C)$ ; the definition is meaningful, for it is proved that the product  $b \cdot h$  does not depend on the choice of the index  $\lambda$ .

**3. Duality theorems.** Let  $r$  and  $s$  be nonnegative integers such that  $r + s = n - 1$ , and let  $C$  and  $D$  be dual groups, with  $C$  compact and  $D$  discrete.

**Theorem 1.** *The groups  $H^r(F, F - U_\lambda; C)$  and  $B^s(U_\lambda - F; D)$  are dual under the multiplication (1), defined as the linking coefficient of cycles taken from the multiplied elements.*

**Proof.** a) The product is continuous. Indeed, by definition,

$$b_\lambda \cdot h_\lambda = \nu(z^s, z^r),$$

where

$$z^s \in b_\lambda \in B^s(U_\lambda - F; D), \quad z^r = \{z_\alpha^r\} \in h_\lambda \in H^r(F, F - U_\lambda; C).$$

Consider  $\Delta z_\alpha^r = x_\alpha^{r-1}$ , where  $x_\alpha^{r-1}$  is an absolute cycle of the set  $F - U_\lambda$ , i.e.  $x_\alpha^{r-1}$  lies in  $S^n - U_\lambda$ . Consequently, there exists a chain  $X_\alpha^r$  such that

$$\Delta X_\alpha^r = x_\alpha^{r-1}.$$

Consider the chain

$$z_\alpha^r - X_\alpha^r = \tilde{z}_\alpha^r.$$

It is clear that  $\tilde{z}_\alpha^r$  is an absolute cycle; therefore

$$\nu(z^s, \tilde{z}_\alpha^r) = \nu(z^s, z_\alpha^r - X_\alpha^r) = \nu(z^s, z_\alpha^r).$$

But  $\nu(z^s, \tilde{z}_\alpha^r)$  is continuous (see (3), p. 465), whence  $b_\lambda \cdot h_\lambda$  is also continuous.

b) If

$$0 \neq b_\lambda \in B^s(U_\lambda - F; D),$$

then there exists an element  $h_\lambda$  of the group  $H^r(F, F - U_\lambda; C)$  such that

$$b_\lambda \cdot h_\lambda \neq 0.$$

Consider  $z^s \in b_\lambda$  and the set

$$F' = F \cup (S^n - U_\lambda).$$

Since  $b_\lambda \neq 0$ , we have

$$z^s \approx 0 \quad \text{in} \quad U_\lambda - F = S^n - F',$$

and, consequently, by Alexander-Pontryagin duality, there exists an absolute cycle

$$\tilde{z}^r = \{\tilde{z}_\alpha^r\}$$

of the set  $F'$ , linked with  $z^s$ . Consider the part of the cycle  $\tilde{z}_\alpha^r$  consisting of those simplexes which have at least one vertex in the set  $F \cap U_\lambda$ . Let  $a$  be a vertex of a simplex lying in  $F \cap U_\lambda$ . Moving the remaining vertices along straight lines connecting them with the point  $a$ , we can ensure that all the remaining vertices lie on the boundary of the set  $F - U_\lambda$ . Thus we obtain a cycle

$$z_\alpha^r \text{ mod } F - U_\lambda.$$

The cycle  $z^r = \{z_\alpha^r\}$  is a relative  $r$ -dimensional cycle of the set  $F \text{ mod } F - U_\lambda$ . Indeed,

$$\tilde{z}_\alpha^r \sim \tilde{z}_\beta^r \quad \text{in } F',$$

therefore there exists a chain  $\varphi^{r+1}$  such that

$$\Delta \varphi^{r+1} = \tilde{z}_\alpha^r - \tilde{z}_\beta^r.$$

The boundary  $\Delta \varphi^{r+1}$  consists of the chain  $z_\alpha^r - z_\beta^r$  and that part of the cycle  $\tilde{z}_\alpha^r - \tilde{z}_\beta^r$  whose simplexes do not meet  $F \cap U_\lambda$ . Consequently,

$$z_\alpha^r \sim z_\beta^r \text{ mod } F - U_\lambda.$$

It is easily proved that the relative cycle  $z^r$  constructed by us remains linked with  $z^s$ . Indeed, the film  $f^{s+1}$  stretched over  $z^s$  may be taken in the open set  $U_\lambda$ , where  $z_\alpha^r$  and  $\tilde{z}_\alpha^r$  coincide. Hence

$$I(f^{s+1}, z_\alpha^r) = I(f^{s+1}, \tilde{z}_\alpha^r) \neq 0.$$

Therefore, for the class  $h_\lambda$  containing the cycle

$$z^r = \{z_\alpha^r\},$$

we have

$$b_\lambda \cdot h_\lambda \neq 0,$$

as was required to prove.

c) If

$$0 \neq h_\lambda \in H^r(F, F - U_\lambda; C),$$

then there exists an element  $b_\lambda$  of the group  $B^s(U_\lambda - F; D)$  such that

$$b_\lambda \cdot h_\lambda \neq 0.$$

Consider the relative cycle

$$z^r = \{z_\alpha^r\} \in h_\lambda.$$

Since  $h_\lambda \neq 0$ , we have

$$z^r \approx 0 \text{ mod } F - U_\lambda.$$

Consider

$$\Delta z_\alpha^r = x_\alpha^{r-1}.$$

The boundary  $x_\alpha^{r-1}$  is an absolute cycle of the set

$$F \cap (S^n - U_\lambda).$$

By virtue of this there exists a chain  $X_\alpha^r$  in  $S^n - U_\lambda$  such that

$$\Delta X_\alpha^r = x_\alpha^{r-1}.$$

Consider

$$z_\alpha^r - X_\alpha^r = \tilde{z}_\alpha^r.$$

It is clear that  $\tilde{z}_\alpha^r$  is an absolute cycle of the set  $F'$ . It is proved that

$$\tilde{z}^r = \{\tilde{z}_\alpha^r\}$$

is an absolute  $r$ -dimensional cycle of the set  $F'$ . Since

$$z^r = \{z_\alpha^r\}$$

is a relative cycle, we have

$$z_\alpha^r \sim z_\beta^r \text{ mod } F - U_\lambda.$$

Consequently, there exists a chain  $c_{\alpha\beta}^{r+1}$  such that

$$\Delta c_{\alpha\beta}^{r+1} = z_{\alpha}^r - z_{\beta}^r + q_{\alpha\beta}^r,$$

where  $q_{\alpha\beta}^r$  is a chain from  $S^n - U_{\lambda}$ . The right-hand side of our equality is an absolute cycle; consequently, the boundary of the right-hand side is equal to zero. Hence

$$-\Delta(z_{\alpha}^r - z_{\beta}^r) = \Delta q_{\alpha\beta}^r$$

or

$$-(x_{\alpha}^{r-1} - x_{\beta}^{r-1}) = \Delta q_{\alpha\beta}^r,$$

i.e.

$$-\Delta(X_{\alpha}^r - X_{\beta}^r) = \Delta q_{\alpha\beta}^r.$$

From the last equality we conclude that

$$X_{\alpha}^r - X_{\beta}^r + q_{\alpha\beta}^r$$

is a cycle of the set  $S^n - U_{\lambda}$ . Let  $A^{r+1}$  be a film,

stretched over  $X_{\alpha}^r - X_{\beta}^r + q_{\alpha\beta}^r$  in  $S^n - U_{\lambda}$ . Consider the boundary of the chain  $c_{\alpha\beta}^{r+1} - A^{r+1}$ :  $\Delta(c_{\alpha\beta}^{r+1} - A^{r+1}) = z_{\alpha}^r - z_{\beta}^r + q_{\alpha\beta}^r - X_{\alpha}^r + X_{\beta}^r - q_{\alpha\beta}^r = (z_{\alpha}^r - X_{\alpha}^r) - (z_{\beta}^r - X_{\beta}^r) = \tilde{z}_{\alpha}^r - \tilde{z}_{\beta}^r$ . Hence we conclude that  $\tilde{z}_{\alpha}^r \sim \tilde{z}_{\beta}^r$  in  $F'$ . Thus we have constructed in  $F'$  a true cycle  $\tilde{z}^r$ . We shall show that this cycle is not homologous to zero in  $F'$ . Suppose the contrary: let there exist in  $F'$  a chain  $\tilde{c}_{\alpha}^{r+1}$  such that  $\Delta\tilde{c}_{\alpha}^{r+1} = \tilde{z}_{\alpha}^r$ . Consider the chain  $c_{\alpha}^{r+1}$ , consisting of that part of the chain  $\tilde{c}_{\alpha}^{r+1}$  whose simplexes have at least one vertex in  $F \cap U_{\lambda}^r$ . Then  $\Delta c_{\alpha}^{r+1} = \Delta\tilde{c}_{\alpha}^{r+1} - \Delta(\tilde{c}_{\alpha}^{r+1} - c_{\alpha}^{r+1}) = \tilde{z}_{\alpha}^r - \Delta(\tilde{c}_{\alpha}^{r+1} - c_{\alpha}^{r+1}) = z_{\alpha}^r + (\tilde{z}_{\alpha}^r - z_{\alpha}^r) - \Delta(\tilde{c}_{\alpha}^{r+1} - c_{\alpha}^{r+1})$ , i.e.  $\Delta c_{\alpha}^{r+1} = z_{\alpha}^r \pmod{F - U_{\lambda}}$ , which contradicts the condition  $h_{\lambda} \neq 0$ .

Thus we have constructed in  $F'$  an absolute  $r$ -dimensional cycle  $\tilde{z}^r = \{\tilde{z}_{\alpha}^r\}$ , not homologous to zero in  $F'$ . Consequently, by the Alexander-Pontryagin duality theorem, in  $S^n - F' = U_{\lambda} - F$  there exists an  $s$ -dimensional cycle  $z^s$  linked with  $\tilde{z}^r$ . Let  $b_{\lambda} \in B^s(U_{\lambda} - F; D)$  contain the cycle  $z^s$ . Then  $b_{\lambda}h_{\lambda} \neq 0$ , as was required to prove. We note that this duality can also be proved by means of the Eilenberg-Steenrod axioms and the Alexander-Pontryagin duality theorem.

**Theorem 2.** The groups  $H^r(A, A - U_{\lambda}; C)$  and  $B^s(U_{\lambda} - A; D)$  are dual under the multiplication (4), defined as the linking coefficient of cycles arbitrarily chosen from the factors.

**Proof.** By Theorem 1, the groups  $B^s(U_{\lambda} - F_a; D)$  and  $H^r(F_a, F_a - U_{\lambda}; C)$  are dual for each  $a$ , and the product of elements  $h_{\lambda}^a \in H^r(F_a, F_a - U_{\lambda}; C)$  and  $b_{\lambda}^a \in B^s(U_{\lambda} - F_a; D)$  is the linking coefficient  $b_{\lambda}^a \cdot h_{\lambda}^a = \nu(z_{\lambda}^{sa}, z_{\lambda}^{ra})$ , where  $z_{\lambda}^{sa} \in b_{\lambda}^a$ ,  $z_{\lambda}^{ra} = \{z_{\alpha}^{ra}\} \in h_{\lambda}^a$ ; moreover, one can show that the homomorphisms  $p_{ab}, q_{ba}$  are conjugate. Consequently, the spectra  $\{H^r(F_a, F_a - U_{\lambda}; C), p_{ab}\}$  and  $\{B^s(U_{\lambda} - F_a; D), q_{ba}\}$  are conjugate. But we have seen that  $H^r(A, A - U_{\lambda}; C) = \lim_{\leftarrow} \{H^r(F_a, F_a - U_{\lambda}; C), p_{ab}\}$ ,  $B^s(U_{\lambda} - A; D) = \lim_{\leftarrow} \{B^s(U_{\lambda} - F_a; D), q_{ba}\}$ .

Therefore the groups  $H^r(A, A - U_\lambda; C)$  and  $B^s(U_\lambda - A; D)$  are dual, as was required to prove.

**Theorem 3.** The groups  $H^r(A, x; C)$  and  $B^s(S^n - A, x; D)$  are dual under the multiplication (5), defined as the linking coefficient of cycles taken from the factors.

**Proof.** By Theorem 2, the groups  $H^r(A, A - U_\lambda; C)$  and  $B^s(U_\lambda - A; D)$  are dual for each  $\lambda$ , and the product of elements  $h_\lambda \in H^r(A, A - U_\lambda; C)$  and  $b_\lambda \in B^s(U_\lambda - A; D)$  is the linking coefficient  $b_\lambda \cdot h_\lambda = b_\lambda^a \cdot h_\lambda^a = \nu(z_\lambda^{sa}, z_\lambda^{ra})$ , where  $b_\lambda^a \in b_\lambda$ ,  $h_\lambda^a \in h_\lambda$ . In addition, the homomorphisms  $\Phi_{\lambda\mu}, F_{\mu\lambda}$ , by Lemma 2, are conjugate; consequently, the spectra  $\{H^r(A, A - U_\lambda; C), \Phi_{\lambda\mu}\}$  and  $\{B^s(U_\lambda - A; D), F_{\mu\lambda}\}$  are conjugate. Since, by definition,

$$H^r(A, x; C) = \varinjlim \{H^r(A, A - U_\lambda; C), \Phi_{\lambda\mu}\},$$

$$B^s(S^n - A, x; D) = \varprojlim \{B^s(U_\lambda - A; D), F_{\mu\lambda}\},$$

it follows that  $H^r(A, x; C)$  and  $B^s(S^n - A, x; D)$  are dual, as was required to prove.

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