ON TANGENTIALLY DEGENERATE SURFACES

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

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(Presented by Academician P. S. Aleksandrov on 11 VI 1960)

An n-dimensional surface V_n of an affine or projective space may have a family of tangent E_n' s depending on r parameters, $0 \leq r \leq n$ ($r = 0$ corresponds to the trivial case of an n-dimensional plane). The number r is called the **rank of the surface**; we call **tangentially degenerate** those surfaces whose rank satisfies the condition $0 < r < n$. Surfaces of rank $r < n$ have been the subject of numerous investigations as analogues of developable surfaces of three-dimensional space and in connection with their significance in the theory of bending of surfaces. In recent years new results in this area have been obtained in the works of M. A. Akivis ([2]) and S. I. Savel' ev ([2,5]). In the work ([3]), devoted to conjugate systems, the author showed that one can construct a surface of rank r by using Peterson transformations of an r-dimensional surface. Here we shall show that such a construction has a general character, i.e. that in this way any surface of rank $r < n$ can be obtained, and from this we shall derive further consequences.

1. Let us make several remarks concerning the correspondence of parallelism between two surfaces of an affine space. We shall say that a surface $\bar{\eta}(u^i)$ of the space E_N is in a correspondence of parallelism (or, more briefly, is **parallel**) with a surface $\bar{\xi}(u^i)$, if the tangent planes at corresponding points of the surfaces are parallel. In doing so we shall assume that the surface $\bar{\xi}(u^i)$ is n-dimensional, but allow degeneration of the surface $\bar{\eta}(u^i)$, which may have dimension $n_1 \leq n$; in particular, it is convenient to regard a fixed point as parallel to any surface. In essence one should distinguish the following possible cases of a parallel correspondence.

1^0. The surface $\bar{\eta}(u^i)$ is a (nondegenerate) Peterson transformation of the surface $\bar{\xi}(u^i)$, i.e. it has dimension n and is not in a homothety or parallel-translation correspondence with the surface $\bar{\xi}(u^i)$.

2^0. The surface $\bar{\eta}(u^i)$ is a degenerate Peterson transformation of the surface $\bar{\xi}(u^i)$, i.e. it has dimension n_1, $1 \leq n_1 < n$, and its tangent E_{n_1} at an arbitrary point is parallel to the tangent E_n of the first surface at all corresponding points. It is easy to give examples of parallelism of this kind. Thus, each of the surfaces

$$\bar{\eta}_1 = \bar{U}(u^1, \dots, u^p)$$

and

$$\bar{\eta}_2 = \bar{V}(v^1, \dots, v^q)$$

is parallel to the surface of translation

$$\bar{\xi} = \bar{U}(u^1, \dots, u^p) + \bar{V}(v^1, \dots, v^q).$$

The curve

$$\bar{\eta} = \bar{V}(v)$$

serves as a degenerate Peterson transformation for the surface

$$\bar{\xi} = \bar{U}(u^i) + \int \varphi(u^i, v)\bar{V}'(v)\, dv.$$

Transformations of parallelism of types 1^0 and 2^0 can be possessed only by surfaces carrying a complete conjugate system (the proof in ([3]) remains valid also in the case of degeneration of $\bar{\eta}(u^i)$). In particular, the dimension of the osculating plane of an n-dimensional surface admitting such a transformation satisfies the condition $\rho \leq n\times$.

$\times(n+1)/2+1$. A sufficient condition for the existence of Peterson transformations of the given surface is the existence on this surface of a completely integrable conjugate system, but conditions that are at once necessary and sufficient are not known to us.

3°. The surface $\bar{\eta}(u^i)$ is homothetic to the given one: $\bar{\eta} = k\bar{\xi} + \bar{c}$, $k = \text{const} \neq 1$, $\bar{c} = \text{const}$. Here we also allow $k = 0$.

4°. The surface is obtained from the given one by parallel translation: $\bar{\eta} = \bar{\xi} + \bar{c}$, $\bar{c} = \text{const}$.

Of course, from the projective point of view the cases 3° and 4° are not distinct.

p. 2. In ([3]) the following method of constructing a tangentially degenerate surface is indicated. Let $\overset{0}{\xi}(u^i)$, $i = 1, 2, \dots, p$, be an arbitrary p-dimensional surface, and let $\overset{1}{\eta}(u^i), \overset{2}{\eta}(u^i), \dots, \overset{q}{\eta}(u^i)$ be q surfaces parallel to it, with the corresponding points $\overset{0}{\xi}, \overset{1}{\eta}, \dots, \overset{q}{\eta}$ determining E_q, intersecting E_p, the tangent plane to the surface $\overset{0}{\xi}$, at one point. Then the surface described by the p-parametric family of these E_q will be a $(p+q)$-dimensional surface of rank $r \leq p$. Setting $\overset{s}{\xi} = \overset{s}{\eta} - \overset{0}{\xi}$, it is convenient to write the equation of such a surface in the form

$$\bar{x}(u^i, v) = \overset{0}{\xi} + \sum_{s=1}^{q} v^s \overset{s}{\xi}. \tag{1}$$

In view of the fact that the tangent plane of the surface $\bar{x}$ is determined by the vectors $\left\{\overset{0}{\xi}_i + \sum_{s=1}^{q} v^s \overset{s}{\xi}_i, \overset{s}{\xi}\right\}$, which, for example, when $v^s = 0$, are linearly independent, it has dimension $n = p + q$. The constancy of its tangent plane at all points of one generator E_q follows immediately from relations of the form $\overset{s}{\eta}_i = a^s_{is}\overset{0}{\xi}_j$, defining the parallelism of the surfaces $\overset{0}{\xi}$ and $\overset{s}{\eta}$. The osculating plane of the surface $\bar{x}$ (at least in a neighborhood of $v^s = 0$) will be spanned by the vectors $\{\overset{s}{\xi}, \overset{0}{\xi}_i, \overset{0}{\xi}_{ij}\}$; its dimension satisfies the inequalities $\rho_0 \le \rho \le q + \rho_0$, where ρ_0 is the dimension of the osculating plane of the surface $\overset{0}{\xi}$, so that $\rho_0 \le p + p(p+1)/2$. Below we shall see that, without loss of generality, one may assume that $\rho = q + \rho_0$. Equation (1), in combination with the known cases in which we know how to find transformations of parallelism of the given surface, cited for example in ([3]), makes it possible to construct concrete examples of tangentially degenerate surfaces; but we shall return to this elsewhere.

If the surface (1) has an osculating plane of dimension $\rho > q + p(p+1)/2 + 1$, then the dimension $\rho_0 > p(p+1)/2 + 1$, and the surface $\overset{0}{\xi}$ has only transformations of parallelism of types 3° and 4°; and then, as is easy to see, $\bar{x}$ is a cone with a $(q-1)$-dimensional vertex (possibly improper).

p. 3. Let us prove the generality of the constructions of p. 2. Let the surface $\bar{x}$ be a surface of rank p (but not $p-1$) and belong to E_N, but not to E_{N-1}. Then

Theorem 1. *Every $(p+q)$-dimensional surface of rank p (but not less than p), lying in E_N (but not in E_{N-1}), can be formed by the motion of an E_q determined by $(q+1)$ points: a point of a p-dimensional surface $\overset{0}{\xi}(u^i)$ and q corresponding points of surfaces $\overset{s}{\eta}(u^i)$, parallel to it. In this case one can always assume that:*

1. The surface $\overset{0}{\xi}(u^i)$ (and hence the surfaces parallel to it) belongs to an E_{N-q} that intersects the generator E_q at one point (in a certain domain of variation of u^i).

2. The dimensions of the osculating planes of the given surface and of $\overset{0}{\xi}$ are related by
$$\rho = \rho_0 + q.$$

3. The surface $\overset{0}{\xi}$ is of rank p.

For the proof it suffices to take $q + 1$ ordinary points of the surface belonging to a fixed generator E_q and not lying in E_{q-1}, and to intersect the surface by subspaces $\overset{s}{E}_{N-q}$, $s = 0, \dots, q$, passing through these points and supplementary to E_q. In a neighborhood of the chosen generator, any of the p-dimensional surfaces obtained in the section may be taken for $\overset{0}{\xi}$, while the others will be parallel to it; moreover, conditions 1-3 will, as is easily verified, be satisfied.

Since in E_{n+1} every n-dimensional surface has transformations of parallelism, in the case of tangentially degenerate hypersurfaces, i.e. under the condition $N = p + q + 1$, $\overset{0}{\xi}$ and $\overset{s}{\eta}$ are arbitrary surfaces lying in parallel E_{N-q}; hence the existence of hypersurfaces of arbitrary rank again follows ([1]).

É. Cartan ([4]) showed that for

$$\rho = p + q + p(p+1)/2$$

a surface of rank p is a cone with a $(q-1)$-dimensional vertex. It is not difficult to strengthen this result. Namely, let

$$\rho > q + p(p+1)/2 + 1.$$

Then the section $\overset{0}{\xi}$ has dimension of the osculating plane

$$\rho_0 > p(p+1)/2 + 1,$$

and all the surfaces $\overset{s}{\eta}$ are similar or equal to it, whence it follows immediately that the surface under consideration is a cone with a $(q-1)$-dimensional (proper or improper) vertex.

Theorem 2. Every $(p+q)$-dimensional surface of rank p with dimension of the osculating plane satisfying the condition

$$\rho > q + p(p+1)/2 + 1$$

is a cone with a $(q-1)$-dimensional vertex.

Without additional restrictions this result cannot be improved. Indeed, consider a curve $\overline{U}(u)$ not lying in E_{q+1}, and construct the developable surface

$$\overline{\xi} = \overline{U} + t^1\overline{U}' + \ldots + t^q\overline{U}^{(q)}.$$

Then take the translation surface

$$\overline{x} = \overline{U}(u) + t^1\overline{U}' + \ldots + t^q\overline{U}^{(q)} + \overline{V}(v', \ldots, v^{p-1}),$$

such that the vectors $\overline{U}', \ldots, \overline{U}^{(q+2)}, \overline{V}_i, \overline{V}_{ij}$ are linearly independent; this surface will be a surface of rank p, and the dimension of its osculating plane will be equal to $q+1+p(p+1)/2$, but it will not be a cone (it is verified that its tangent planes contain not a single point common to them all, proper or improper). Hence it is seen that Theorem 1 in ([2]) is formulated incorrectly.

From the same considerations as in the case of Theorem 2, we obtain

Theorem 3. If a $(p + q)$-dimensional surface of rank p admits at least one section by a plane E_{N-q} that does not carry a complete conjugate system, then it is a cone with a $(q-1)$-dimensional vertex.

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Received
27 V 1960

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Note: Figure translations are in progress. See original paper for figures.