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Abstract

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Yu. I. ZHURAVLEV

ON ALGORITHMS FOR SIMPLIFYING DISJUNCTIVE NORMAL FORMS

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Unambiguous algorithms for simplifying a reduced disjunctive normal form (d.n.f.) are based on deleting from $\mathfrak{N}$ conjunctions for which it has been established that they are not contained in the d.n.f. $\mathfrak{N}_{\Sigma M}$* ([1–3]). There exist various criteria for the non-inclusion of a conjunction $\mathfrak{A}$ from $\mathfrak{N}$ in $\mathfrak{N}_{\Sigma M}$; all of them are based on studying conjunctions close to $\mathfrak{A}$. By close conjunctions one may understand, for example, conjunctions $\mathfrak{A}_i$ from $\mathfrak{N}$ such that $\mathfrak{A}_i \cdot \mathfrak{A} \neq 0$. In the process of searching for conjunctions not belonging to $\mathfrak{N}_{\Sigma M}$, it is sometimes possible to obtain additional information about some conjunctions from $\mathfrak{N}$. Thus Quine' s method ([1,2]) makes it possible to distinguish certain conjunctions that enter into all minimal d.n.f.' s; subsequently this helps in the search for conjunctions not belonging to $\mathfrak{N}_{\Sigma M}$. It is natural to regard unambiguous algorithms for simplifying a d.n.f. $\mathfrak{N}$ as algorithms that place marks above the conjunctions from $\mathfrak{N}$. Marks of the first type are placed above conjunctions not belonging to $\mathfrak{N}_{\Sigma M}$, and marks of the second type above conjunctions belonging to $\mathfrak{N}_{\Sigma M}$. After the completion of the algorithm that places marks, all conjunctions with marks of the first type are deleted from $\mathfrak{N}$. At this point the simplification algorithm ends. We now pass to a formal description of algorithms for simplifying d.n.f.' s.

1°. Together with conjunctions $\mathfrak{A}$ we shall consider conjunctions with marks $\mathfrak{A}^{(j)}$ $(j = 1, 2, 3, 4)$. Unmarked conjunctions $\mathfrak{A}$ will be denoted by $\mathfrak{A}^{(0)}$. Conjunctions $\mathfrak{A}_l^{(l)}$ and $\mathfrak{B}_j^{(m)}$ such that $\mathfrak{A}_i = \mathfrak{B}_j$, $l \neq m$, are, in essence, identical conjunctions for which, in the course of the algorithm, different information has been obtained. We shall call conjunctions $\mathfrak{A}^{(i)}$ and $\mathfrak{B}^{(j)}$ equal with respect to information if $\mathfrak{A} = \mathfrak{B}$ and $i = j$.

2°. We shall consider d.n.f.' s composed of conjunctions with marks $(0), (1), (2), (3), (4)$, satisfying the following conditions:
1) if $\mathfrak{A}^{(i)} \subseteq \mathfrak{N}$ and $i \neq j$, then $\mathfrak{A}^{(j)} \nsubseteq \mathfrak{N}$;
2) if $\mathfrak{A}^{(1)} \subseteq \mathfrak{N}$, then $\mathfrak{A} \subseteq \mathfrak{N}_{\Sigma M}$;
3) if $\mathfrak{A}^{(2)} \subseteq \mathfrak{N}$, then $\mathfrak{A} \nsubseteq \mathfrak{N}_{\Sigma M}$;
4) if $\mathfrak{A}^{(3)} \subseteq \mathfrak{N}$, then $\mathfrak{A} \subseteq \mathfrak{N}_{\Sigma M}$ and $\mathfrak{A} \nsubseteq \mathfrak{N}_{\cap M}$**;
5) if $\mathfrak{A}^{(4)} \subseteq \mathfrak{N}$, then $\mathfrak{A} \subseteq \mathfrak{N}_{\cap M}$.
D.n.f.' s composed of conjunctions $\mathfrak{A}^{(i)}$, $(i = 0, 1, 2, 3, 4)$, and satisfying

conditions 1)–5), will be called *admissible*. In what follows we shall consider only admissible d.n.f.' s. With each d.n.f. $\mathfrak{N}$ we associate the sets $M(\mathfrak{N})$, $M^{(j)}(\mathfrak{N})$ $(j = 0, 1, 2, 3, 4)$. $M(\mathfrak{N})$ is composed of all conjunctions contained in $\mathfrak{N}$, with the marks above the conjunctions erased. $M^{(j)}(\mathfrak{N})$ is composed of all conjunctions contained in $\mathfrak{N}$ with mark (j). D.n.f.' s $\mathfrak{N}_1$ and $\mathfrak{N}_2$ will be called *equal with respect to information* if

$$M^{(j)}(\mathfrak{N}_1) = M^{(j)}(\mathfrak{N}_2) \quad (j = 0, 1, 2, 3, 4).$$

* The d.n.f. $\mathfrak{N}_{\Sigma M}$ is composed of all conjunctions from $\mathfrak{N}$ that enter into some minimal d.n.f.

** $\mathfrak{N}_{\cap M}$ is composed of all conjunctions that enter into all d.n.f.' s minimal with respect to $\mathfrak{N}$.

An algorithm for simplifying a D.N.F. is completely determined by specifying an algorithm that assigns marks, and an algorithm that determines the order in which marks are assigned. The algorithm that assigns marks reduces to the computation of a function φ, to the description of which we now turn.

1°. **Domain of definition.** To each pair $(\mathfrak{A}^{(j)}, \mathfrak{N})$, where $\mathfrak{A}^{(j)} \subseteq \mathfrak{N}$, we assign a set $S(\mathfrak{A}^{(j)}, \mathfrak{N})$, consisting of conjunctions occurring in $\mathfrak{N}$, with $\mathfrak{A}^{(j)} \in S(\mathfrak{A}^{(j)}, \mathfrak{N})$. The set $S(\mathfrak{A}^{(j)}, \mathfrak{N})$ will be called the **neighborhood** of $\mathfrak{A}^{(j)}$ in $\mathfrak{N}$, and $\mathfrak{A}^{(j)}$ will be called the **center of the neighborhood**. To each neighborhood $S(\mathfrak{A}^{(j)}, \mathfrak{N})$ we assign the sets $M[S(\mathfrak{A}^{(j)}, \mathfrak{N})]$, $M^{(i)}[S(\mathfrak{A}^{(j)}, \mathfrak{N})]$ $(i = 0, 1, 2, 3, 4)$. The set $M[S(\mathfrak{A}^{(j)}, \mathfrak{N})]$ is composed of all conjunctions occurring in S, with the marks over the conjunctions erased. $M^{(i)}[S(\mathfrak{A}^{(j)}, \mathfrak{N})]$ is composed of all conjunctions with mark (i) occurring in $S(\mathfrak{A}^{(j)}, \mathfrak{N})$. We shall consider systems of neighborhoods for which the following condition is satisfied: if $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are such that $M[S(\mathfrak{A}, \mathfrak{N}_1)] \subseteq M(\mathfrak{N}_2)$ and $M(\mathfrak{N}_2) \subseteq M(\mathfrak{N}_1)$, then $M[S(\mathfrak{A}, \mathfrak{N}_1)] = M[S(\mathfrak{A}, \mathfrak{N}_2)]$. The neighborhoods $S(\mathfrak{A}^{(j)}, \mathfrak{N}_1)$ and $S(\mathfrak{B}^{(i)}, \mathfrak{N}_2)$ will be called **equal in information** if their centers are equal in information and $M^{(k)}[S(\mathfrak{A}^{(j)}, \mathfrak{N}_1)] = M^{(k)}[S(\mathfrak{B}^{(i)}, \mathfrak{N}_2)]$ $(k = 0, 1, 2, 3, 4)$. The domain of definition of the function φ is the system of all neighborhoods.

2°. The range of values of φ consists of the marks (0), (1), (2), (3), (4). On neighborhoods that are not equal in information, the values of φ, generally speaking, are different.

Let us introduce a partial ordering: 1) in the set of marks: $(j) = (j)$, $(0) < (j)$ $(j = 1, 2, 3, 4)$; $(1) < (j)$ $(j = 3, 4)$; 2) in the set of marked conjunctions: $\mathfrak{A}^{(i)} \preccurlyeq \mathfrak{B}^{(j)}$, if $\mathfrak{A} = \mathfrak{B}$, $(i) \leq j$; 3) in the set of neighborhoods: $S(\mathfrak{A}^{(i)}, \mathfrak{N}_1) \preccurlyeq S(\mathfrak{B}^{(j)}, \mathfrak{N}_2)$, if: a) $\mathfrak{A}^{(i)} \preccurlyeq \mathfrak{B}^{(j)}$, b) $M[S(\mathfrak{A}^{(i)}, \mathfrak{N}_1)] = M[S(\mathfrak{B}^{(j)}, \mathfrak{N}_2)]$, c) for every pair of identical conjunctions $\mathfrak{A}^{(k)}$ and $\mathfrak{A}^{(l)}$ such that $\mathfrak{A}^{(k)} \in S(\mathfrak{A}^{(i)}, \mathfrak{N}_1)$, $\mathfrak{A}^{(l)} \in S(\mathfrak{B}^{(j)}, \mathfrak{N}_2)$, the relation $\mathfrak{A}^{(k)} \preccurlyeq \mathfrak{A}^{(l)}$ holds; 4) in the set of D.N.F.s: $\mathfrak{N}_1 \preccurlyeq \mathfrak{N}_2$, if: a) $M(\mathfrak{N}_1) = M(\mathfrak{N}_2)$, b) for every pair of identical conjunctions such that $\mathfrak{A}^{(j)} \subseteq M(\mathfrak{N}_1)$, $\mathfrak{A}^{(i)} \subseteq M(\mathfrak{N}_2)$, the relation $\mathfrak{A}^{(j)} \preccurlyeq \mathfrak{A}^{(i)}$ holds. It is obvious that if $\mathfrak{N}_1 \preccurlyeq \mathfrak{N}_2$ and $\mathfrak{N}_2 \preccurlyeq \mathfrak{N}_1$, then $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are equal in information.

Definition. The function φ is called **monotone** if, for any pair of neighborhoods $S_1 = S_1(\mathfrak{A}^{(i)}, \mathfrak{N}_1)$, $S_2 = S_2(\mathfrak{A}^{(j)}, \mathfrak{N}_2)$ such that $S_1 \preccurlyeq S_2$, the inequality $\varphi(S_1) \leq \varphi(S_2)$ holds.

We shall study algorithms with monotone φ. Algorithms with nonmonotone φ will be considered later. We shall decompose D.N.F.-simplification algorithms into elementary algorithms and logical conditions ([4,5]), describe their functioning, and write down a logical scheme ([4,5]), common to all simplification algorithms with monotone function φ. Elementary algorithms: 1) A_π selects in $\mathfrak{N}$ all conjunctions belonging to the set $M^0(\mathfrak{N}) \cup M^{(1)}(\mathfrak{N})$, and orders this set, with the order being established identically for D.N.F.s $\mathfrak{N}_1$ and $\mathfrak{N}_2$ such that $M^{(i)}(\mathfrak{N}_1) = M^{(i)}(\mathfrak{N}_2)$ $(i = 0, 1)$; 2) $A(i)$ selects in $M^0(\mathfrak{N}) \cup M^{(1)}(\mathfrak{N})$ the element $\mathfrak{A}_i^{(j)}$ with number i and the neighborhood $S(\mathfrak{A}_i^{(j)}, \mathfrak{N})$; 3) $A_\varphi(i)$ computes $\varphi[S(\mathfrak{A}_i^{(j)}, \mathfrak{N})] = \tilde{\varphi}$; 4) $B_\varphi(i)$ transforms the conjunction $\mathfrak{A}_i^{(j)}$ into the conjunction $\mathfrak{A}^{\tilde{\varphi}}$; 5) $F(i+1)$ adds one to i; 6) $F(i \to 1)$ transforms i into one; 7) C deletes from $\mathfrak{N}$ all conjunctions with mark (2); 8) the symbol ω indicates the end of the algorithm. Logical conditions p_1, $p_2(i)$, p_3: $p_1 = 0$, if $M^{(0)}(\mathfrak{N}) \cup M^{(1)}(\mathfrak{N})$ is empty; $p_1 = 1$, if $M^{(0)}(\mathfrak{N}) \cup M^{(1)}(\mathfrak{N})$ is nonempty; $p_2(i) = 1$, if $\varphi[S(\mathfrak{A}_i^{(j)}, \mathfrak{N})] = \tilde{\varphi} = (j)$; $p_2(i) = 0$, if $\tilde{\varphi} \neq (j)$. Let k be the number of the last element in $M^{(0)}(\mathfrak{N}) \cup M^{(1)}(\mathfrak{N})$. $p_3(i) = 0$, if $i - 1 < k$; $p_3(i) = 1$, if $i - 1 = k$.

Simplification algorithms process the reduced DNF $\mathfrak{N}$, composed of unmarked conjunctions, into a DNF composed of conjunctions with marks, and then delete from $\mathfrak{N}$ the conjunctions with mark (2). The logical scheme of the simplification algorithm has the form (here φ is a monotone function)

$$\underset{2}{\perp}\, F(i \to 1) A_\pi p_1 \underset{1}{\perp}\underset{3}{\perp}\, A(i) A_\varphi(i) F(i+1) \underset{2}{\perp}\, p_3 \underset{3}{\perp}\underset{1}{\perp}\, C\omega. \qquad (1)$$

There exist DNF simplification algorithms that do not fit into the scheme described. These algorithms do not guarantee obtaining minimal DNFs under further simplifications. On the other hand, the simplification algorithms known to us, in whose execution no loss of minimal DNFs occurs, fit into the scheme described by us.

To each simplification algorithm with a logical scheme of the form (1) one may put into one-to-one correspondence a pair (A_π, φ). We shall say that algorithms A and B **belong to the same class** $K(\varphi)$ if the pairs $((A_\pi)', \varphi_1)$ and $((A_\pi)'', \varphi_2)$ assigned to them are such that $\varphi_1 = \varphi_2$. We shall call the elements of the class $K(\varphi)$ **algorithms generated by the function** φ. We shall call simplification algorithms A and B **equivalent** if the DNFs $A(\mathfrak{N})$ and $B(\mathfrak{N})$, obtained after applying A and B to an arbitrary reduced DNF $\mathfrak{N}$ composed of unmarked conjunctions, are equal in information. To every algorithm A and DNF $\mathfrak{N}$ one may assign a sequence $\mathfrak{N}_1, \dots, \mathfrak{N}_\alpha$ (denoted $C(\mathfrak{N}, A)$) possessing the following properties: 1) the algorithm A successively transforms $\mathfrak{N}$ into $\mathfrak{N}_1, \dots, \mathfrak{N}_{\alpha-1}$, into $\mathfrak{N}_\alpha$; 2) neighboring elements of the sequence differ by marks over exactly one conjunction; 3) $A(\mathfrak{N})$ coincides with the DNF obtained from $\mathfrak{N}_\alpha$ by deleting all conjunctions with mark (2); 4) $\mathfrak{N}_1 \leq \mathfrak{N}_2 \leq \cdots \leq \mathfrak{N}_\alpha$ (by virtue of the monotonicity of φ). We shall call $\mathfrak{N}_\alpha$ the **image of $\mathfrak{N}$ in the algorithm** A. It is obvious that, for equivalence of the algorithms A and B, it is necessary and sufficient that the images of an arbitrary DNF $\mathfrak{N}$, composed of unmarked conjunctions, in the algorithms A and B coincide.

Algorithms with a monotone function φ possess an interesting uniqueness property: the final result of applying the algorithm does not depend on A_π.

Theorem. *If the elements of the class $K(\varphi)$ are generated by a monotone function, then all algorithms from $K(\varphi)$ are equivalent.*

Proof. Let

$$\mathfrak{N} = \bigvee_{i=1}^{m} \mathfrak{A}_i^{(0)}.$$

1) Assign to the DNF $\mathfrak{N}$ a sequence of DNFs $\mathfrak{N}, \mathfrak{N}^{(1)}, \dots, \mathfrak{N}^{(\alpha)}$. Let

$$\varphi[S(\mathfrak{A}_i^{(0)}, \mathfrak{N})] = \varphi_{1i} \quad (i = 1, 2, \dots, m).$$

Put

$$\mathfrak{N}^{(1)} = \bigvee_{i=1}^{m} \mathfrak{A}_i^{\varphi_{1i}}.$$

If $\mathfrak{N}$ and $\mathfrak{N}^{(1)}$ are equal in information, then assign to $\mathfrak{N}$ the sequence consisting of one element $\mathfrak{N}$.

2) Suppose the elements of the sequence $\mathfrak{N}, \mathfrak{N}^{(1)}, \dots, \mathfrak{N}^{(k-1)}$ have been formed and all of them are distinct in information. Let

$$\mathfrak{N}^{(k-1)} = \bigvee_{i=1}^{m} \mathfrak{A}^{\varphi_{k-1,i}}.$$

Denote

$$\varphi[S(\mathfrak{A}^{\varphi_{k-1,i}}, \mathfrak{N}^{(k-1)})]$$

by φ_{ki}. Put

$$\mathfrak{N}^{(k)} = \bigvee_{i=1}^{m} \mathfrak{A}^{\varphi_{ki}}.$$

If $\mathfrak{N}^{(k-1)}$ and $\mathfrak{N}^{(k)}$ are equal in information, then assign to $\mathfrak{N}$ the sequence

$$\mathfrak{N}, \mathfrak{N}^{(1)}, \dots, \mathfrak{N}^{(k-1)}.$$

By virtue of the monotonicity of φ, in the construction described there will be found an index S such that

$$\mathfrak{N}^{(S)} = \mathfrak{N}^{(S+1)}.$$

The sequence will have the form

$$\mathfrak{N}, \mathfrak{N}_1, \dots, \mathfrak{N}^{(S)}.$$

Here

$$\mathfrak{N}^{(S)} = \bigvee_{i=1}^{m} \mathfrak{A}^{\varphi_{Si}}$$

and

$$\mathfrak{N} \leq \mathfrak{N}^{(1)} \leq \dots \leq \mathfrak{N}^{(S)}.$$

We shall show that the image $\mathfrak{N}_A$ of the DNF $\mathfrak{N}$ in an arbitrary algorithm A, generated by the function φ, is equal in information to $\mathfrak{N}^{(S)}$.

I. We indicated that to the algorithm A one can associate the sequence $C(\mathfrak{N}, A) = \{\mathfrak{N}, \mathfrak{N}_1, \dots, \mathfrak{N}_\alpha\}$ (here $\mathfrak{N}_\alpha = \mathfrak{N}_A$). Let us show that $\mathfrak{N}_\alpha \preceq \mathfrak{N}^{(s)}$. We carry out induction on $C(\mathfrak{N}, A)$. The induction scheme is: 1) $\mathfrak{N}^{(1)} \preceq \mathfrak{N}^{(s)}$; 2) from the fact that $\mathfrak{N}^{(k-1)} \preceq \mathfrak{N}^{(s)}$, it follows that $\mathfrak{N}^{(k)} \preceq \mathfrak{N}^{(s)}$.

II. $\mathfrak{N}_\alpha \succeq \mathfrak{N}^{(s)}$. The induction scheme is: 1) in the sequence $C(\mathfrak{N}, A)$ there is found an element $\mathfrak{N}_{k_1}$ such that $\mathfrak{N}_{k_1} \succeq \mathfrak{N}^{(1)}$; 2) suppose that in $C(\mathfrak{N}, A)$ there are found DNFs $\mathfrak{N}_{k_i}$ and $\mathfrak{N}_{k_j} \succeq \mathfrak{N}^{(i)}$. Then in $C(\mathfrak{N}, A)$ there is found a DNF $\mathfrak{N}_{k_{j+1}}$, and $\mathfrak{N}_{k_{j+1}} \succeq \mathfrak{N}^{(j+1)}$.

From $\mathfrak{N}_A \preceq \mathfrak{N}^{(s)}$ and $\mathfrak{N}_A \succeq \mathfrak{N}^{(s)}$ we conclude that $\mathfrak{N}_A$ and $\mathfrak{N}^{(s)}$ are equal in information. The theorem is proved.

To describe the logical scheme of simplification algorithms with a nonmonotone function φ, we modify the definition of the elementary algorithm A_π. The algorithm A_π orders the set $M(\mathfrak{N})$, and, if the DNFs $\mathfrak{N}_1$ and $\mathfrak{N}_2$ are equal in information, then $M(\mathfrak{N}_1)$ and $M(\mathfrak{N}_2)$ are ordered identically. In the description of the other elementary algorithms and logical conditions, instead of the set $M^{(0)}(\mathfrak{N}) \cup M^{(1)}(\mathfrak{N})$ one should consider the set $M(\mathfrak{N})$. To an algorithm A

generated by a nonmonotone function φ with a logical scheme of the form (1) one can associate an infinite, generally speaking, sequence $\mathfrak{N}, \mathfrak{N}_1, \ldots, \mathfrak{N}_\alpha, \ldots$ such that: 1) A transforms successively $\mathfrak{N}$ into $\mathfrak{N}_1, \ldots, \mathfrak{N}_{\alpha-1}$ into $\mathfrak{N}_\alpha, \ldots$; 2) neighboring elements differ by marks exactly over one conjunction; 3) beginning with some $\mathfrak{N}_\gamma$ ($\gamma = \gamma(A, \mathfrak{N})$), the sequence becomes periodic (in the sense of equality in information). After obtaining $\mathfrak{N}_\gamma$, it is natural to terminate the algorithm. Introduce the elementary algorithms: 1) P forms the sequence $\mathfrak{N}, \mathfrak{N}_1, \ldots, \mathfrak{N}_\alpha, \ldots$ as it is obtained by the algorithm A. At the first step P forms the element $\mathfrak{N}$; 2) C' selects, in the elements of the sequence formed by the algorithm P, the conjunctions with mark (2) and deletes them from the corresponding DNFs. The logical condition $p_4(\mathfrak{N}, \mathfrak{N}_1, \ldots, \mathfrak{N}_\alpha) = 1$ if none of the DNFs $\mathfrak{N}, \mathfrak{N}_1, \ldots, \mathfrak{N}_{\alpha-1}$ is equal to $\mathfrak{N}_\alpha$ in information. Otherwise $p_4(\mathfrak{N}, \mathfrak{N}_1, \ldots, \mathfrak{N}_\alpha) = 0$; $p_4(\mathfrak{N}) = 1$. The logical scheme of an algorithm generated by the nonmonotone function φ has the form:

$$\tfrac{1}{2}|\, F(i \to 1)\, P p_{4\;4} \,|\, A_{\pi\;3} \,|\, A(i)\, A_\varphi(i)\, B_\varphi(i)\, F(i+1)\, p_{2\;2} \,|\, p_{3\;3} \,|\;_4 |\, C'\omega.$$

We shall call the DNFs $\mathfrak{N}_1, \ldots, \mathfrak{N}_\gamma$ **images** of the DNF $\mathfrak{N}$ in the algorithm A.

Moscow State University
named after M. V. Lomonosov

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Note: Figure translations are in progress. See original paper for figures.