

# REALIZATION OF A TWO-DIMENSIONAL METRIZED MANIFOLD BY A SURFACE IN $\mathbb{E}^3$

1960

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**Abstract**

**Full Text**

**MATHEMATICS**

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## **REALIZATION OF A TWO-DIMENSIONAL METRIZED MANIFOLD BY A SURFACE IN $E^3$**

*(Presented by Academician V. I. Smirnov, 4 VII 1960)*

1. In the works of J. Nash <sup>(1)</sup> and N. Kuiper <sup>(2)</sup>, an isometric embedding of an  $n$ -dimensional Riemannian space  $R^n$  into  $(n+1)$ -dimensional Euclidean space  $E^{n+1}$  was carried out. In these works the metric in  $R^n$  is given by means of a quadratic form with coefficients of class  $C^1$ .

In the case of a two-dimensional manifold, in the present paper a realization is constructed for more general metrics, given by a function of a pair of points. From these results it follows, in particular, that the geometry of any manifold of bounded curvature in the sense of A. D. Aleksandrov <sup>(3)</sup> can be regarded as the intrinsic geometry of a certain surface in  $E^3$ .

2. Let an intrinsic metric  $\rho$  be given in a manifold  $M$ , and let there be a sequence of polyhedral metrics  $\rho_n$ . The metrics  $\rho_n$  are called proportionally convergent to  $\rho$  if

$$\frac{\rho_n(X, Y)}{\rho(X, Y)} \rightarrow 1 \quad \text{as } n \rightarrow \infty$$

uniformly with respect to  $X, Y$ .

If the metric  $\rho$  has a tangent cone at each point (in the sense of intrinsic geometry), then, as Yu. G. Reshetnyak proved <sup>(4)</sup>,  $\rho$  is approximated by proportionally convergent polyhedral metrics.

Denote by  $\Phi$  the class of non-self-intersecting surfaces in  $E^3$  at each point of which there exists a tangent plane. (These surfaces, generally speaking, are not smooth.)

**Theorem.** *Let the metric  $\rho$ , given on a two-dimensional orientable manifold  $M$ , admit an approximation by proportionally convergent polyhedral metrics  $\rho_n$ . Then there exists a surface  $F \in \Phi$  with intrinsic metric  $\rho$ .*

**Corollary 1.** *Let at each point of a two-dimensional orientable manifold  $M$  with metric  $\rho$  there exist a tangent cone. Then there exists a surface  $F \in \Phi$  with intrinsic metric  $\rho$ .*

**Corollary 2.** Every orientable manifold of bounded curvature, having no points with curvature  $2\pi$ , with boundary in the form of a curve of bounded variation of rotation or without boundary, is isometric to a certain surface  $F \in \Phi$ .

3. We outline the plan of the proof of the theorem. Let  $\rho$  be an arbitrary metric satisfying the condition of the theorem; let  $\rho_n$  be polyhedral metrics proportionally convergent to  $\rho$ . With each metric  $\rho_n$  we associate a certain topological mapping  $\varphi_n$  of the manifold  $M$  onto itself.

**Lemma 1.** *The sequence of metrics  $\rho_n$  and mappings  $\varphi_n$  can be chosen so that:*

- 1)  $\rho_{n+1}(\varphi_n(X), \varphi_n(Y)) > \rho_n(X, Y)$ ;
- 2)  $\varphi_n \in C^\infty(D_n)$ ,  $D_n = M \setminus \bigcup A_i$ , where  $A_i$  are the vertices of all metrics  $\rho_k$  for  $k \leq n$ ;
- 3)  $\rho_{n+1}(X, \varphi_n(X)) \rightarrow 0$  uniformly with respect to  $X$ ;

4)

$$\frac{\rho_{n+1}(\varphi_n(X), \varphi_n^*(Y))}{\rho(X, Y)} \rightarrow 1$$

uniformly with respect to  $X, Y$ , and the sequences 3)–4) converge just as rapidly as  $\rho_n(X, Y) \rightarrow \rho(X, Y)$ .

The mappings  $\varphi_n$  are constructed by means of triangulations of the manifold  $M$ . These constructions are based on the following:

**Lemma 2.** *Suppose the total angles about the vertices and the angles between adjacent edges of the boundary of the development  $K$  are bounded below by a number  $\alpha > 0$ . Then the development  $K$  can be divided into  $n$  plane triangles adjacent along whole sides, in such a way that all angles of the resulting triangles are bounded below by a number  $C(\alpha) > 0$ , depending only on  $\alpha$ .*

The proof of Lemma 2 is similar to the proof of Theorem 2 in <sup>(5)</sup>.

With the aid of Caper' s construction, each metric  $\rho_n$  (beginning with some sufficiently large  $n$ ) is embedded in  $E^3$  as a surface  $F_n \in C^\infty(D_n)$ . The surface  $F_n \in \Phi$  has a metric close to  $\rho_n$  and is situated near the surface  $F_{n-1}$ , which, in view of 1)–2), serves as a “short” embedding for  $\rho_n$  (see <sup>(2)</sup>). With a suitable choice of the parameters in Caper' s formulas for the surfaces  $F_n$ , the desired surface  $F$ , realizing the metric  $\rho$ , is the limit of the sequence of surfaces  $F_n$ . The proof of this fact is based on properties 2)–4) of the metrics  $\rho_n$ .

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Received  
16 VI 1960

## REFERENCES CITED

<sup>1</sup> J. Nash, *Sbornik perevodov Matematika*, 1, Nos. 2, 3 (1957). <sup>2</sup> N. Caper, *ibid.*, p. 17. <sup>3</sup> A. D. Aleksandrov, *DAN*, 60, No. 9 (1948). <sup>4</sup> Yu. G. Reshetnyak, *Izv. Sibirsk. otd. AN SSSR*, 10, 15 (1959). <sup>5</sup> Yu. D. Burago, V. A. Zalgaller, *Vestn. LGU*, 7, 66 (1960).

*Note: Figure translations are in progress. See original paper for figures.*

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