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Abstract

Full Text

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The Riemann Boundary-Value Problem with a Measurable Coefficient

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1°. Let C be a contour consisting of $m+1$ simple closed Lyapunov contours $C_0, C_1, \ldots, C_m$, bounding a connected domain D^+. The complementary part of the plane, consisting of the sum of m finite simply connected domains D_k^+ and the infinite domain D_0^-, will be denoted by D^-. The class of functions summable with degree p (>1) on the contour C will, as usual, be denoted by $L_p(C)$. If a function $\varphi^\pm(z)$ analytic in $D^\pm$ is representable by a Cauchy integral and $\varphi^\pm(t) \in L_p(C)$, then we shall write $\varphi(z) \in F_p(D^\pm)$.

We formulate the Riemann problem as follows:

Find functions $\Phi^\pm(z)$, belonging to $E_2(D^\pm)$ and satisfying on C the boundary condition

$$\Phi^+(t) = G(t)\Phi^-(t) + g(t), \tag{1}$$

where $g(t) \in L_2$.

The Riemann problem in the classical formulation, when $G(t)$ and $g(t)$ satisfy a Hölder condition, was solved by F. D. Gakhov (1). A further generalization to the case of a multiply connected domain (2) and g summable with degree p (3) is due to B. V. Khvedelidze. The case of a discontinuous coefficient was considered by S. G. Mikhlin (4), I. Ts. Gokhberg (5), B. V. Khvedelidze (6), V. V. Ivanov (7), and the author (8). In the present paper we dispense with the continuity condition on the function $G(t)$, with respect to which the following is assumed: 1) $G(t)$ is a measurable function; 2) $0 < M_1 \leq |G(t)| \leq M_2 < \infty$; 3) there exists a δ (>0) such that for every point t_0 of the contour C there is a neighborhood in which the values $G(t)$ are contained in an angle with vertex at the origin and opening $\pi - \delta$. Property 3) means that the oscillation of the argument of $G(t)$ at each point is less than $\pi - \delta$. This condition replaces continuity.

The class of functions $G(t)$ satisfying conditions 1), 2), and 3) will be called class A. It turns out that even under such an extension of the class of coefficients, F. D. Gakhov' s theorems concerning the number of solutions and solvability conditions remain valid.

2°. Properties of class A. Applying the Heine-Borel theorem, it is easy to see that the following condition is fulfilled:

I. There exists a finite covering (B) of the contour C by intervals, on each of which the values $G(t)$ are contained in an angle of opening $\pi - \delta$ with vertex at the origin.

On each contour C_k take a point t_k and cut it at this point, i.e., regard the point t_k as two points. From the covering (B) of the uncut contours C_k form a covering of the cut contours C_k, replacing intervals containing t_k by half-segments with closed ends t_k.

Assigning the value $\arg G(t)$ at the points t_k and proceeding in the direction of traversal of the contour along the chain of intervals, we shall successively determine $\arg G(t)$ on the intervals encountered so that $|\arg G(t) - \arg G(t')| \leq \pi$ when t and t' belong to the same interval. As a result we obtain a completely

a certain branch $\varphi(t)$ of the argument of $G(t)$, which at the points t_k has two values: one $\varphi(t_k^-)$ before the circuit, the other $\varphi(t_k^+)$ after the circuit. It is easy to see that $\varphi(t)$ depends on t_k and does not depend on the choice of the covering (B), provided only that it satisfies condition I. The difference

$$\frac{1}{2\pi}\left[\varphi(t_k^+) - \varphi(t_k^-)\right] = \varkappa_k \tag{2}$$

does not depend even on the choice of t_k. The sum

$$\sum_{k=0}^{m} \varkappa_k = \varkappa = \operatorname{Ind} G \tag{3}$$

will be called the **index of the problem**.

It can be proved that functions of class A have the following property: every function of class A can be represented in the form

$$G(t) = G_1(t) G_2(t), \tag{4}$$

where $G_1(t)$ satisfies the Hölder condition, vanishes nowhere, and $\operatorname{Ind} G_1(t) = \varkappa = \operatorname{Ind} G(t)$; $G_2(t)$ belongs to class A and $|\arg G_2(t)| \leqslant \dfrac{\pi - \delta_1}{2}$ for some constant $\delta_1 > 0$.

On the basis of the representation (4), by the method set forth in the work ([8]), the original problem can be reduced to the problem with coefficient G_2

$$\Phi^+ = G_2 \Phi^- + g. \tag{5}$$

3°. Problem (5) is equivalent to the problem

$$\Phi_{1\gamma}^{+} = \gamma G_2 \Phi_1^{-} + g, \tag{6}$$

where γ ($\neq 0$) is a constant. The resulting problem can be regarded as a singular integral equation with respect to the function Φ_1, related to $\Phi_1^{\pm}$ by the relations

$$\Phi_1^{\pm} = \pm\frac{1}{2}\Phi_1 + \frac{1}{2\pi i}\int_C \frac{\Phi_1(\tau)}{\tau - t}\,d\tau, \qquad \Phi_1 = \Phi_1^{+} - \Phi_1^{-}.$$

On the basis of what has been said, equation (6) can be rewritten as follows:

$$\Phi_1 = (\gamma G_2 - 1)\Phi_1^{-} + g. \tag{7}$$

By transforming the contour, consisting of $m+1$ nonintersecting unit circles $\Gamma_0, \Gamma_1, \dots, \Gamma_m$, into the contour C, equation (7) can be reduced to an equation on the contour $\Gamma_0 + \Gamma_1 + \dots + \Gamma_m$:

$$\Phi_2(t) = (\gamma G_2 - 1)\left[-\frac{1}{2}\Phi_2(t) + \frac{1}{2\pi i}\sum_{k=0}^{m} P_k \int_{\Gamma_k} \frac{\Phi_2(\tau)}{\tau - t}\,d\tau\right] + K\Phi_2, \tag{8}$$

where P_k is the truncation operator:

$$P_k\varphi(t) = \begin{cases} \varphi(t), & t \in \Gamma_k, \\ 0, & t \notin \Gamma_k; \end{cases}$$

K is a completely continuous operator.

By choosing γ ($\neq 0$), the operator

$$(\gamma G_2 - 1)\left[-\frac{1}{2}\Phi_2 + \frac{1}{2\pi i}\sum_{k=0}^{m} P_k \left\{\int_{\Gamma_k} \frac{\Phi_2(\tau)}{\tau - t}\,d\tau\right\}\right]$$

can be made a contraction operator in the space $L_2(\Gamma_0 + \Gamma_1 + \dots + \Gamma_m)$, whence it follows that equation (8), and along with it problem (5), satisfy the following condition:

II. They have a finite number of linearly independent solutions, coinciding with the number of solvability conditions.

Let us show that the homogeneous problem has only the trivial solution. We shall obtain this by representing $G_2(t)$ in the form

$$G_2(t) = \frac{X^{+}(t)}{X^{-}(t)}, \tag{9}$$

where

$$X(z) = \exp\left[\frac{1}{2\pi i}\int_C \frac{\ln G_2(\tau)}{\tau - z}\,d\tau\right],$$

and $(X^{\pm})^{\pm 1} \in E_2(D^{\pm})$.

The latter is proved by a conformal mapping of the domain $D_k^{\pm}$ onto a disk and by using, for the disk, the method of V. I. Smirnov ([9]).

Having the representation (9), the homogeneous problem (5) can be rewritten in the form

$$\frac{\Phi^+}{X^+} = \frac{\Phi^-}{X^-},$$

and, since both sides are representable by a Cauchy integral, one concludes that $\Phi^{\pm} = 0$. Together with II this makes it possible to draw the conclusion:

Problem (5) is unconditionally and uniquely solvable.

4°. Using the conclusion obtained and the method set forth in ([8]), we obtain the following proposition, which constitutes the main result of the paper.

Theorem. *In the case $\chi = 0$, the Riemann boundary value problem (1) is solvable and has a unique solution. In the case $\chi > 0$, the problem is also unconditionally solvable and has χ linearly independent components of the general solution. In the case $\chi < 0$, the problem is solvable only when $|\chi|$ solvability conditions are satisfied, and then has a unique solution.*

The theorem remains valid also for the Riemann–Haseman problem

$$\Phi^+[\alpha(t)] = G(t)\Phi^-(t) + g(t),$$

where $\alpha(t)$ maps the contour C one-to-one onto itself with preservation of the direction of traversal, and $\alpha'(t)$ satisfies the Hölder condition.

The theorem proved permits one to generalize many results on boundary value problems for analytic functions and related questions to the case of coefficients from the class A.

Corollary 1. Suppose that a singular integral equation is given:

$$a(t)\varphi(t) + \frac{b(t)}{\pi i}\int_C \frac{\varphi(\tau)}{\tau - t}\,d\tau + K\varphi = g(t); \tag{10}$$

$a(t)$ and $b(t)$ satisfy the condition

$$G(t) = \frac{a(t) - b(t)}{a(t) + b(t)} \in A;$$

K is a completely continuous operator in $L_2(C)$; φ and $g \in L_2(C)$.

Then for equation (10) the Noether theorems ([1]), p. 178, hold with index $\chi = \operatorname{Ind} G$ (in the sense of § 2°).

In what follows we shall assume the domain D^+ to be simply connected.

Corollary 2. Consider the Hilbert problem

$$\operatorname{Re} G(t)W(t) = g(t), \tag{11}$$

where $W(z)\,(\in E_2(D^+))$ is the unknown function; $g(t) \in L_2(C)$; $G(t) \in A$, and at each point the local oscillation of the argument of $G(t)$ is less than $\pi/2$*. Then for problem (11) the theorem from ([1]), p. 233, is valid. In this case $\chi = -\operatorname{Ind} G$.

* This requirement is dictated by the fact that the Hilbert problem (11) is reduced to the Riemann problem (1) with coefficient $G_3 = G/\overline{G}$, as a result of which the oscillation of the argument of G_3 is doubled in comparison with the oscillation of the argument of G.

Corollary 3. Consider the problem with oblique derivative

$$a(t)\frac{\partial u}{\partial x} + b(t)\frac{\partial u}{\partial y} + c(t)u = f(t), \qquad t \in C, \tag{12}$$

for a harmonic function u in the class $\frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \in E_2(D^+)$. With respect to the given functions it is assumed that $f(t) \in L_2(C)$; $c(t) \in L_2(C)$; $a(t) + ib(t)\,(\in A)$ has at each point t a local oscillation of the argument $< \pi/2$. Then, for the problem so formulated, the known theorems are valid (([10], p. 225). In this connection the index should everywhere be understood in the sense of item 2°.

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REFERENCES

[1] F. D. Gakhov, *Boundary Value Problems*, 1958.
[2] B. V. Khvedelidze, *Reports of the Academy of Sciences of the Georgian SSR*, **2**, No. 7, 10 (1941).
[3] B. V. Khvedelidze, *Proceedings of the Tbilisi Mathematical Institute*, **23** (1956 –1957).

[4] S. G. Mikhlin, *Doklady AN*, **59**, No. 3 (1948).
[5] I. Ts. Gokhberg, *Doklady AN*, **122**, No. 3 (1958).
[6] B. V. Khvedelidze, *Reports of the Academy of Sciences of the Georgian SSR*, **21**, No. 2 (1958).
[7] V. V. Ivanov, *Doklady AN*, **121**, No. 5 (1953).
[8] I. B. Simonenko, *Doklady AN*, **124**, No. 2 (1959).
[9] V. Smirnoff, *Math. Ann.*, **107**, 313 (1932).
[10] N. I. Muskhelishvili, *Singular Integral Equations*, 1946.

Note: Figure translations are in progress. See original paper for figures.