ON ERGODIC CLASSES OF RECURRENT MOTIONS

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Abstract

Full Text

MATHEMATICS

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ON ERGODIC CLASSES OF RECURRENT MOTIONS

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If a set of motions is described by Bohr almost-periodic functions ([1]), then it is strictly ergodic ([2]). A. A. Markov ([3]) showed that the presence of recurrent motions ([4]) of a general character, generally speaking, violates strict ergodicity. In the present paper the existence is shown of a number of classes of recurrent motions possessing the ergodic property. Here the construction of such classes is carried out in the set of all continuous functions given for $t \in (-\infty, +\infty)$, without reference to any dynamical system.

Definition 1. A continuous function $f(t)$, given for $t \in (-\infty, +\infty)$, is called **recurrent** if for every $\varepsilon > 0$ and every real t one can indicate a number L_ε such that in every interval of the real axis of length L_ε there exists at least one number $\tau_{t,\varepsilon}$ such that the inequality

$$|f(t+\tau_{t,\varepsilon}) - f(t)| < \varepsilon. \tag{1}$$

is satisfied.

Theorem 1. *Every recurrent function is bounded, and the totality of all such functions forms a complete space in the sense of uniform convergence on the entire real axis.*

The main difficulty in creating a universal mathematical apparatus for representing recurrent functions consists in the fact that the concept of recurrence is not invariant with respect to addition and, moreover, is not transitive in the sense that if $c_1 f_1 + c_2 f_2$ and $d_1 f_2 + d_2 f_3$ are recurrent functions, this still does not mean that $l_1 f_1 + l_2 f_3$ will also be a recurrent function; here c_i, d_i, l_i are arbitrary constants.

Definition 2. The set of all recurrent functions $f(t)$ satisfying the condition: for every $\varepsilon > 0$ one can indicate a finite collection of forms with real coefficients, linear with respect to $\gamma_1(t), \dots, \gamma_N(t)$, $p_{l\varepsilon}$, $l \le k_\varepsilon$, and a number $\delta > 0$ such that all jointly real solutions of the system of inequalities

$$|p_{l\varepsilon}(t+\tau) - p_{l\varepsilon}(t)| < \delta \pmod{2\pi}, \qquad l \le k_\varepsilon, \tag{2}$$

for any fixed t satisfy inequality (1), will be called the class $H(\gamma_1(t), \dots, \gamma_N(t))$, where $\gamma_1(t), \dots, \gamma_N(t)$ are arbitrary real continuous functions, given for $t \in (-\infty, +\infty)$, such that uniformly in $c \in (-\infty, +\infty)$

$$\lim_{T\to\infty} \frac{1}{T} \int_c^{c+T} \exp\left[i \sum_{k=1}^{N} \lambda_k \gamma_k(t) \right] dt = 0 \tag{3}$$

for any real numbers $\lambda_1, \dots, \lambda_N$ such that

$$\sum_{k=1}^{N} \lambda_k \gamma_k(t) \not\equiv 0.$$

Of course, the number δ and the forms $p_{l\varepsilon}$ themselves depend, generally speaking, on the specifically chosen function $f(t)$.

Let us note that condition (3) ensures the existence of a relatively dense set of common solutions of system (2) and is certainly satisfied when, for example, $\gamma_j(t) = t^j$.

Theorem 2. For every function $f(t) \in H(\gamma_1(t), \dots, \gamma_N(t))$ one can indicate the largest countable number of linear forms with respect to $\gamma_1, \dots, \gamma_N$, with real coefficients $p_1(t), p_2(t)$, such that for $\varepsilon > 0$ one can indicate numbers k_ε and $\delta > 0$ such that all common real solutions of the system of inequalities

$$|p_k(t+\tau) - p_k(t)| < \delta \pmod{2\pi}, \qquad k = 1, \dots, k_\varepsilon, \tag{4}$$

for any fixed t, satisfy inequality (1).

Corollary. If $N = 1$ and $\gamma_1(t) = t$, then from Theorem 2, on the basis of N. N. Bogolyubov' s theorem ([4]), one may conclude that $H(t)$ is the totality of all Bohr almost-periodic functions. In connection with this, the problem arises of determining the class $H(\gamma_1(t), \dots, \gamma_N(t))$ through the arithmetical properties of the almost-periods of the function $f(t)$ in such a way as, with the aid of some analogue of the mentioned theorem of N. N. Bogolyubov, to arrive at the property of the class $H(\gamma_1(t), \dots, \gamma_N(t))$ taken here as the basis of Definition 2.

Theorem 3. The class H_N is a complete linear space in the sense of uniform convergence on the whole real axis (here and below H_N denotes the class $H(\gamma_1(t), \dots, \gamma_N(t))$, when the specific form of the functions $\gamma_1(t), \dots, \gamma_N(t)$ is immaterial).

Corollary. Put

$$H_\infty = \overline{\bigcup_{N=1}^{\infty} H(\gamma_1(t), \dots, \gamma_N(t))}.$$

It too will be a complete linear space.

Theorem 4. If $f(t) \in H_N$, then for each $\varepsilon > 0$ one can indicate a trigonometric polynomial

$$p_\varepsilon = \sum_{k=1}^{N_\varepsilon} c_k e^{i\pi_k(t)} \quad N_\varepsilon < \infty,$$

such that

$$|f(t) - p_\varepsilon(t)| < \varepsilon,$$

where $\pi_k(t)$ are finite linear forms with integer coefficients of those functions $p_1(t), p_2(t), \ldots$ which are referred to in Theorem 2.

The assertion of Theorem 4 remains valid also in the space H_∞, if one observes that in this case the functions $p(t)$ will be linear forms of $\gamma_1, \ldots, \gamma_{N_k}$, $N_k \xrightarrow[k\to\infty]{} \infty$.

Theorem 5. For any function $f(t) \in H_N$ or $f(t) \in H_\infty$ there exist mean values

$$M_t(f(t)) = \lim_{T\to\infty} \frac{1}{T} \int_0^T f(t)\, dt, \tag{5}$$

and, moreover,

$$\frac{1}{T} \int_c^{c+T} \gamma(t)\, dt \xrightarrow[T\to\infty]{} M_t(f(t))$$

uniformly with respect to $c \in (-\infty, +\infty)$.

Remark. The existence of a mean value for functions of the class H_N also means the existence of the ergodic property.

Definition 3. If

$$A(\Lambda) = M_t\left(f(t) \exp\left[-i \sum_{j=1}^{N} \lambda_j \gamma_j(t)\right]\right) \neq 0,$$

then

$A(\Lambda)$ is called the **Fourier coefficient** of the function $f(t)$ in the class H_N, and the real vectors $(\lambda_1, \ldots, \lambda_N)$ are called the **Fourier exponents** of this function in the class H_W, or simply the Fourier coefficients and Fourier exponents of the function $f(t)$.

Theorem 6. *Every function $f \in H_N$ has at most a countable set of Fourier exponents and determines its Fourier series uniquely,*

$$f \sim \sum_{k=1}^{\infty} c_k e^{ip_k(t)}$$

in such a way that Parseval's equality is satisfied

$$M_t(|f(t)|^2) = \sum_{k=1}^{N} |c_k|.$$

Let us note that an analogous theorem is valid for H_∞, with some obvious changes. In both cases the Fourier series are summable, and their sum coincides with the generating function.

Definition 4. The function

$$K_{kmn_1 \dots n_m} = \prod_{l=1}^{m} K_{ln_l}(x,t)$$

will be called the **generalized Fejér kernel** of the function $f(t) \in H_N$, where

$$K_{ln_l} = \sin^2\left[\frac{1}{2} n_l \left(\frac{q_l(x) - q_l(t)}{k}\right)\right] \left(n_l \sin^2 \frac{1}{2}\left(\frac{q_l(x) - q_l(t)}{k}\right)\right)^{-1},$$

$$q_l = \sum_{j=1}^{N} y_{lj} \gamma_j(t).$$

Here the system of vectors $Y_1, Y_2, \dots$ $(Y_l = (y_{l1} \dots y_{Nl}))$ is chosen so that it forms a rational basis of the entire aggregate of Fourier exponents of the function $f(t)$.

Definition 5. The trigonometric polynomial

$$p_{kmn_1 \dots n_m} = M_t(f(t) \cdot K_{kmn_1 \dots n_m}(x,t))$$

is called the **generalized trigonometric Fejér sum**.

Theorem 7. *Whatever the function $f \in H_N$ may be, there exists a sequence of the corresponding Fejér sums $S_k(t)$ such that*

$$S_k(t) \xrightarrow[k \to \infty]{} f(t)$$

uniformly with respect to $t \in (-\infty, +\infty)$.

Theorem 8. *Every function $f \in H_N$ can be represented in terms of the limit-periodic (1) function $F(z_1, \dots, z_N)$ by the formula*

$$f(t) = F(\gamma_1(t)E, \gamma_2(t)E, \dots, \gamma_N(t)E),$$

where z_i is a countably dimensional vector with real coordinates; E is a countably dimensional vector with unit coordinates.

Let us note that some generalizations of the class H_N are possible; apparently one should here be guided by the known generalizations of Bohr almost-periodic functions made by Besicovitch, V. V. Stepanov, André Weil, and B. Ya. Levitan (6).

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CITED LITERATURE

1. H. Bohr, *Acta math.*, **45** (1925).
2. V. V. Nemytskii, V. V. Stepanov, *Qualitative Theory of Differential Equations*, 2nd ed., Moscow-Leningrad, 1949.
3. A. A. Markov, *Matem. sborn.*, **4**(46), 165 (1938).
4. N. N. Bogolyubov, *Zap. kaf. matem. fiziki AN USSR*, **4**, 185 (1939).
5. V. V. Stepanov, *DAN*, **64**, No. 3 (1949).
6. A. Besicovitch, *Proc. London Math. Soc.* (2), **25** (1926).
7. B. M. Levitan, *Almost-Periodic Functions*, Moscow, 1953.

Note: Figure translations are in progress. See original paper for figures.