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Abstract

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ON THE STATISTICAL THEORY OF A NON-LINEAR FIELD

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1. Let us consider a set of classical fields $\psi_1(x_0, x_1, x_2, x_3)$, $\psi_2(x_0, x_1, x_2, x_3)$, ..., $\psi_N(x_0, x_1, x_2, x_3)$, whose equations of motion are determined from the principle of least action

$$S = \int L\left[\psi_1,\ \psi_2,\ \dots,\ \psi_N;\ \frac{\partial\psi_1}{\partial x_0},\ \frac{\partial\psi_1}{\partial x_1},\ \dots,\ \frac{\partial\psi_N}{\partial x_3}\right] dx_0\, dx_1\, dx_2\, dx_3 = S_m, \quad (1)$$

where L is the Lagrangian of the system, and S_m is the extremal value of the action S. In a statistical consideration of this set of fields, to each possible field one must assign a certain a priori probability or, more precisely, a probability density in functional space.

The simplest probability distribution, forbidding fields that do not satisfy the equations of motion and assigning equal-probability values to all possible solutions of the equations of motion, is evidently the distribution

$$W = A(S_m)\, \delta[S_m - S\{\psi, \partial\psi/\partial x\}], \quad (2)$$

where δ is the Dirac delta function; ψ and $\partial\psi/\partial x$ are abbreviated notations for the totality of all fields and their derivatives; A is a normalization constant, which, generally speaking, is a functional, since S_m depends on the total energy and on various integral invariants (for example, total charge, etc.). By analogy with classical statistical mechanics, distribution (2) may also be called **microcanonical**.

The mean values of any physical quantities $F\{\psi(x)\}$ are determined, evidently, by the functional integral*

$$\overline{F} = \int F\{\psi(x)\} W\{\psi(x)\}\, d\psi(x). \quad (3)$$

Here and below, by x we mean the totality of all four Minkowski coordinates $x_0,\ x_1,\ x_2,\ x_3$; we use braces only for the arguments of functionals, while for the arguments of functions we shall use only parentheses or square brackets.

The probability that a certain field function $Q[\psi(x)]$ has the prescribed form $q(x)$ is determined, evidently, by the integral

$$W\{q(x)\} = \int \delta\{q(x) - Q[\psi(x)]\} W\{\psi(x)\}\, d\psi(x), \tag{4}$$

where $\delta\{y(x)\}$ is a delta-functional, defined by the equation

$$F\{\eta(x)\} = \int \delta\{y(x) - \eta(x)\} F\{y(x)\}\, dy(x). \tag{5}$$

Evidently, this functional may be defined as

$$\delta\{y(x)\} = \int \left[\exp i \int 2\pi\xi(x) y(x)\, dx\right] d\xi(x). \tag{6}$$

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* For the functional integral we use here the notation introduced by us in 1945 in a doctoral dissertation, published in abbreviated form in 1949 [1].

2. If some aggregate of the fields $\chi_\nu(x)$, from among those considered above (or, perhaps, some class of solutions of the equations of motion), can be assigned to a "thermostat," i.e., regarded as quantities ignored in the given physical problem, then for the remaining fields $\varphi(x)$, which are of interest in this problem, one may introduce the canonical distribution:

$$W = B \exp[bS\{\varphi(x)\}], \tag{7}$$

where b is a numerical constant, and B is a normalizing constant which is a function of b and of the parameters entering into S. Just as in statistical mechanics, the distribution (7) can be obtained from the distribution (2) under certain natural simplifying assumptions.

3. If the aggregate of fields $\varphi(x)$ that interests us obeys linear equations of motion, i.e., if the Lagrangian L is a quadratic function of these fields and their derivatives, then the distribution (7) leads to a paradox called the "ultraviolet catastrophe," consisting in the fact that the field energy, even in a closed volume, tends to infinity as a result of its uniform distribution over monochromatic waves of all frequencies.

4. The situation is different in the nonlinear case, when the Lagrangian is, for example, a sum of quadratic terms and terms of the 4th, 6th, and higher orders. In this case, for some class of Lagrangians, among physically meaningful solutions with finite energy, finite charge, etc., there exist particle-like solutions (see, for example, ([2])), representing stable field clumps (packets). For such nonlinear Lagrangians, solutions in the form of monochromatic waves of small amplitude are, of course, also possible,

since at small amplitudes the nonlinear field equations pass over into linear ones. However, the solutions existing in the linear case in the form of a sum of monochromatic waves of all frequencies of approximately equal amplitude* are forbidden in this nonlinear case, since such solutions also imply delta-shaped, short-time condensations of the field of arbitrarily large amplitude at the point of maximum. But the latter solutions are impossible, since with finite energy in our nonlinear problem only particle-like solutions with a bounded maximum value of the field are possible ([2]).

If we consider, in general, the case of a field that is not spatially bounded in any way and regard its energy and certain integral invariants as finite, then only particle-like solutions have physical significance. All other solutions have the form of quasilinear wave packets, which rapidly spread through all space and whose amplitude thus tends to zero.

Thus, for nonlinear Lagrangians admitting particle-like solutions, when the conditions of finiteness of energy and of certain integral invariants are imposed, all fields admissible by the principle of least action will be practically concentrated in the neighborhoods of world lines depicting the motion of particle-like maxima. But in this case the action functional of the field can be represented as a sum of integrals over the aggregate of world lines of certain functionals of the path, which in mechanics are also called action functionals. That is, one may put:

$$S\{x(t)\} = \int L[x_1(t), \dot{x}_1(t); x_2(t); \dot{x}_2(t), ...]dt, \tag{8}$$

where $x_k(t)$ is a shortened notation for the three spatial coordinates as functions of time, depicting world line number k, and $x(t)$ denotes all the world lines.

* These solutions are the cause of the "ultraviolet catastrophe," since if there were a restriction of the spectrum from the side of small wavelengths, there would be no divergence.

5. Obviously, for such nonlinear fields the probability density can be represented not as a functional of the entire field, but as a functional of the world lines depicting the motion of the maxima of particle-like solutions. According to (2) and (8) we obtain the microcanonical distribution

$$W\{x(t)\} = A\delta[S_m - S\{x(t)\}], \tag{9}$$

and according to (4) the canonical one

$$W\{x(t)\} = B\exp[bS\{x(t)\}] \tag{10}$$

as the distribution for a system in a "thermostat." The mean value of any quantity $F\{x(t)\}$, obviously, is determined according to the formula

$$\bar{F} = \int F\{x(t)\} W\{x(t)\}\, dx(t). \tag{11}$$

By an analogous formula one can, obviously, also calculate the probability of a given value of any physical quantity at a given instant of time.

From expression (10) it is not difficult to obtain, for example, the probability density of a given configuration of N material points $q_1, q_2, \dots, q_{3N}$. For this it is necessary to integrate (10) over the entire set of world lines $x(t)$ passing through the given configuration of points at a given instant of time. In the nonrelativistic approximation the Lagrangian of the system is equal to the difference between the kinetic and potential energies of the system. Substituting such a Lagrangian into (10) and integrating over all $x(t_k)$, except the given $x(t_0)$, we obtain

$$W(q_1, q_2, \dots, q_{3N}) = \exp[\tau \cdot b[\Psi - U]], \tag{12}$$

where U is the potential energy of the system; τ is the minimal time interval which must be introduced in the practical implementation of functional integration; Ψ is a constant determined from the normalization condition. If we set $\tau \cdot b = 1/\Theta$, where $\Theta = kT$, i.e. Boltzmann' s constant multiplied by the temperature, then expression (12) is equivalent to the usual canonical Gibbs distribution.

6. Of particular interest is the special case of the distribution (10) with an imaginary value of the constant b. If such a distribution is written in the form

$$W\{x(t)\} = B \exp\left[\frac{i}{\hbar} S\{x(t)\}\right], \tag{13}$$

then it is easy to see that the mean values of physical quantities calculated by formula (11) with distribution (13) coincide with the quantum-mechanical averages of these quantities, provided that $\hbar$ is taken equal to Planck' s constant divided by 2π.

Indeed, in the nonrelativistic case the integral of expression (13) over all space-time paths arriving at a given point from all preceding time points is equal to the wave function $\Psi(q_1, q_2, \dots, q_{3N}, t)$, as was shown by Feynman ([3]), and the integral of (13) over all paths leaving this point for all subsequent time points is equal to $\Psi^*(q_1, q_2, \dots, q_{3N}, t)$. But, in accordance with our rules (10), (11), expression (13) must be integrated over all paths arriving from $t = -\infty$ and leaving at $t = +\infty$, i.e. the probability of a given configuration is equal to the product $\Psi\Psi^*$, i.e. is equal to the quantum-mechanical probability.

In an analogous manner, as was shown by G. V. Ryazanov ([4]), one can calculate the probability density of momenta and of other physical quantities by taking (13) as the probability of given paths arriving from $t = -\infty$ and leaving at

$t = +\infty$. Here it is essential that the physical quantities F, whose probabilities and mean values are calculated according to (11), (13), must be regarded as classical mechanical quantities.

Thus, all results of quantum theory are obtained as consequences of the general statistical theory of a nonlinear classical field,

having particle-like solutions, if one assumes that this field is in statistical equilibrium with a thermostat having a constant imaginary temperature (since the constant b can be interpreted as a quantity inversely proportional to the temperature).

7. However, two important questions arise: 1) What physical meaning can an imaginary temperature have? 2) What ensures the constancy of the imaginary temperature?

Let us consider a gas of relativistic particles of identical mass m, in equilibrium with a thermostat. The canonical distribution for such a gas is evidently written in the form

$$W(\mathbf{p}_1, \mathbf{p}_2, \dots, \mathbf{p}_N) = \exp\left[\frac{\Psi - M}{\theta}\right], \tag{14}$$

where

$$M = \left[\left(\sum_k \varepsilon_k\right)^2 - \left(\sum_k \mathbf{p}_k\right)^2\right]^{1/2}$$

(i.e., the total rest mass of the system); $\mathbf{p}_k$ are the momenta of the particles; $\varepsilon_k = (\mathbf{p}_k^2 + m^2)^{1/2}$ are their energies (a system of units is chosen in which $c = 1$). For ordinary particles the four-dimensional vector $\mathbf{p}_k, \varepsilon_k$ is timelike, and consequently m and M are real quantities. One may, however, imagine particles with a spacelike four-dimensional momentum. For such particles the mass m will be imaginary, and consequently M may also turn out to be imaginary. But in this case, for real θ, W will be complex, and the mean values of the momenta will also turn out to be complex. Moreover, for infinite values of the momenta the probability density will not tend to zero. Consequently, expression (14) cannot be normalized and, thus, in general the distribution (14) loses its usual physical meaning. If, however, for particles with imaginary proper mass one also takes θ to be imaginary, then the distribution (14) regains its usual physical meaning.

The question of the admissibility of particles with imaginary mass was discussed by us specially in the preceding article ([5]). It was found that the theory of relativity and the second law of thermodynamics do not prohibit such particles; however, these particles cannot carry negative entropy in the same way as particles with real mass. The assumption of an "imaginary" thermostat also explains the constancy of the imaginary temperature. Indeed, according to ([5]), particles of imaginary mass cannot carry entropy. Consequently, the entropy of a real system interacting with an "imaginary" thermostat cannot change, whatever the energy of the real system may be. Thus, the assumption that the world around us is filled with particles of imaginary mass with a constant imaginary temperature is a physically admissible and internally consistent hypothesis.

8. Thus, all quantum effects may be regarded as the result of the interaction of a classical system with real proper mass with an "imaginary" thermostat, i.e., with a system consisting of particles of imaginary mass and being in a state of statistical equilibrium. Planck' s constant thereby acquires the meaning of a quantity proportional to the temperature of the "imaginary" thermostat, multiplied by a certain constant having the dimension of time.

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Note: Figure translations are in progress. See original paper for figures.