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Abstract

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GEOPHYSICS

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ON ONE CLOSED-FORM SOLUTION OF THE TURBULENT DIFFUSION EQUATION

(Presented by Academician A. A. Dorodnitsyn on 12 X 1959)

The turbulent diffusion equation

$$\frac{\partial q}{\partial t} + u_x \frac{\partial q}{\partial x} + u_z \frac{\partial q}{\partial z} = \frac{\partial}{\partial x}\left(k_x \frac{\partial q}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial q}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial q}{\partial z}\right) \tag{1}$$

under the assumption that $u_x, k_x, k_y, k_z = \text{const}$,

$$u_z = 2k_z\sqrt{a}\ \mathrm{tg}\, z\sqrt{a}, \quad 0 \le z \le H; \qquad u = \text{const}, \quad H \le z \le \infty,$$

is solved with the following initial and boundary conditions:

$$\text{for } t = 0 \quad q = Q\delta(x)\delta(y)\delta(z-h);$$

$$\text{for } z = 0 \quad k_z\, \partial q/\partial z - wq = 0; \qquad \text{as } \sqrt{x^2+y^2+z^2} \to \infty \quad q = 0$$

(xy is the ground plane; z is the vertical coordinate; u_x, u_z are the corresponding components of wind velocity; k_x, k_y, k_z are the corresponding coefficients of turbulent diffusion; Q is the amount of substance released at the initial instant of time from the source $x = y = 0,\ z = h$; q is the volume concentration).

It is known that the solution of equation (1), under the above initial and homogeneous boundary conditions, can be represented in the form

$$q = q_x(x,t)q_y(y,t)q_z(z,t).$$

In this case we shall have

$$\left.\begin{aligned} &\frac{\partial q_z}{\partial t} + 2k_z\sqrt{a}\ \mathrm{tg}\, z\sqrt{a}\,\frac{\partial q_z}{\partial z} = k_z \frac{\partial^2 q_z}{\partial z^2}, \\ &\frac{\partial q_x}{\partial t} + u_x \frac{\partial q_x}{\partial x} = k_x \frac{\partial^2 q_x}{\partial x^2}, \qquad \frac{\partial q_y}{\partial t} = k_y \frac{\partial^2 q_y}{\partial y^2}, \end{aligned}\right\} \quad 0 \le z \le H; \tag{2}$$

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$$\left.\begin{aligned}&\frac{\partial q_z}{\partial t}+u_z\frac{\partial q_z}{\partial z}=k_z\frac{\partial^2 q_z}{\partial z^2},\\&\frac{\partial q_x}{\partial t}+u_x\frac{\partial q_x}{\partial x}=k_x\frac{\partial^2 q_x}{\partial x^2},\qquad \frac{\partial q_y}{\partial t}=k_y\frac{\partial^2 q_y}{\partial y^2},\end{aligned}\right\}\quad H\le z\le\infty. \tag{3}$$

The solution of the second and third equations with constant coefficients in systems (2), (3) is trivial. In the article only the solution for q_z is given. Thus, the equations

$$\frac{\partial q_{z\mathrm{I}}}{\partial t}+2k_z\sqrt{a}\ \mathrm{tg}\, z\sqrt{a}\,\frac{\partial q_{z\mathrm{I}}}{\partial z}=k_z\frac{\partial^2 q_{z\mathrm{I}}}{\partial z^2},\qquad 0\le z\le H; \tag{4}$$

$$\frac{\partial q_{z\mathrm{II}}}{\partial t}+u_z\frac{\partial q_{z\mathrm{II}}}{\partial z}=k_z\frac{\partial^2 q_{z\mathrm{II}}}{\partial z^2},\qquad H\le z\le\infty, \tag{5}$$

are solved under the following initial and boundary conditions:

$$\text{for } t=0\qquad q_{z\mathrm{I}}=\delta(z-h),\qquad q_{z\mathrm{II}}=0,\qquad h\le H,$$

or

$$q_{z\mathrm{I}}=0,\qquad q_{z\mathrm{II}}=\delta(z-h),\qquad h\ge H,$$

$$\lim\left(k_z\frac{\partial q_{z\mathrm{I}}}{\partial z}-u_z q_{z\mathrm{I}}\right)_{z=0}=0,\qquad \frac{\partial q_{z\mathrm{I}}}{\partial z}=\frac{\partial q_{z\mathrm{II}}}{\partial z},\qquad q_{z\mathrm{I}}=q_{z\mathrm{II}}\quad\text{for } z=H;$$

$$\text{for } \sqrt{x^2+y^2+z^2}\to\infty\qquad q_{z\mathrm{II}}=0.$$

Equations (4), (5) are solved by the operational method. We multiply both sides of equations (4), (5) by $e^{-p_t t}$ and integrate them, respectively, with respect to t from 0 to ∞. Taking the initial condition into account, we obtain:

$$\frac{d^2\tilde q_{z\mathrm{I}}}{dz^2}-2\sqrt{a}\ \mathrm{tg}\, z\sqrt{a}\,\frac{d\tilde q_{z\mathrm{I}}}{dz}-\frac{p_t}{k_z}\tilde q_{z\mathrm{I}}=-\frac{\delta(z-h)}{k_z}, \tag{6}$$

$$\frac{d^2\tilde q_{z\mathrm{II}}}{dz^2}-\frac{u_z}{k_z}\frac{d\tilde q_{z\mathrm{II}}}{dz}-\frac{p_t}{k_z}\tilde q_{z\mathrm{II}}=0,\qquad h\ll H. \tag{7}$$

As a result of solving equations (6) and (7) and determining the corresponding constants from the boundary conditions written above, we obtain that in the lower layer

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Fig. 1

Figure 1: Fig. 1

Fig. 1

$$q_{z\mathrm{I}} = \frac{\cos h\sqrt{a}}{k_z \pi i \cos z\sqrt{a}} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{\lambda t}\ \mathrm{ch}(H-h)\sqrt{\frac{\lambda}{k_z} - a}\ \mathrm{ch}\, z\sqrt{\frac{\lambda}{k_z} - a} \times$$

$$\times \frac{\left[\frac{\sqrt{u_z^2 + 4\lambda k_z}}{4k_z\sqrt{\frac{\lambda}{k_z} - a}}\ \mathrm{th}(H-h)\sqrt{\frac{\lambda}{k_z} - a} + \frac{1}{2}\right]}{\mathrm{ch}\, H\sqrt{\frac{\lambda}{k_z} - a}\ \frac{\sqrt{u_z^2 + 4\lambda k_z}}{2k_z} + \sqrt{\frac{\lambda}{k_z} - a}\ \mathrm{sh}\, H\sqrt{\frac{\lambda}{k_z} - a}}\, d\lambda. \tag{8}$$

The integrand in (8) has a single pole at the point $\lambda/k_z = 0$, which leads to the trivial solution $q_{z\mathrm{I}} = \mathrm{const}$. $\lambda/k_z = a$ is a removable singular point. The required solution is obtained by evaluating integral (8) with respect to the branch point $\lambda = -u_z^2/4k_z$.

To evaluate integral (8), it is sufficient to choose the contour shown in Fig. 1. The integral will be equal to the sum of the integrals along the upper and lower cuts.

In (8) let us make the change of variable $\lambda = (\lambda^* - u_z^2)/4k_z$. Put $\lambda^* = \rho e^{i\pi}$ for the upper cut and $\lambda^* = \rho e^{-i\pi}$ for the lower cut. We multiply the numerator and denominator of the integrand by

$$\cos H\sqrt{\frac{\rho + u_z^2}{4k_z^2} + a}.$$

After a number of simple transformations, we obtain

$$q_{z\mathrm{I}} = \frac{\cos h\sqrt{a}\ \exp\left[-\frac{u_z^2}{4k_z}t\right]}{2k_z\pi\cos z\sqrt{a}} \times$$

$$\times \int_0^\infty \frac{\sqrt{\rho}\ \exp\left[-\frac{\rho}{4k_z}t\right] \cos z\sqrt{\frac{\rho + u_z^2}{4k_z^2} + a}\ \cos h\sqrt{\frac{\rho + u_z^2}{4k_z^2} + a}}{\rho + (u_z^2 + 4k_z^2 a)\sin^2 H\sqrt{\frac{\rho + u_z^2}{k_z^2} + a}}\, d\rho. \tag{9}$$

It is easy to show that the solution (9), obtained by the operational method, satisfies equation (4) and the boundary condition $\left(k_z\frac{\partial q_{z1}}{\partial z} - u_z q_{z1}\right)_{z=0} = 0$. In order to show that it also satisfies the initial condition, we write it in the form

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$$q_{z1} = \frac{\exp\left[-\frac{u_z^2}{4k_z}t\right]\cos h\sqrt{a}}{4k_z\pi\cos z\sqrt{a}} \times \int_0^\infty \frac{\sqrt{\rho}\exp\left[-\frac{\rho}{4k_z}t\right]\left[\cos(z-h)\sqrt{\alpha\rho+\beta}+\cos(z+h)\sqrt{\alpha\rho+\beta}\right]}{\rho + A\sin^2 H\sqrt{\alpha\rho+\beta}}\,d\rho. \tag{10}$$

Here $\alpha = 1/4k_z^2$, $\beta = u_z^2/4k_z^2 + a$; $A = u_z^2 + 4k_z^2 a$; $\beta = A\alpha$.

Making the substitution $\frac{1}{2}\sqrt{\frac{\rho t}{k_z}} = u$, we obtain

$$q_{z1} = \frac{\exp\left[-\frac{u_z^2}{4k_z}t\right]\cos h\sqrt{a}}{\pi\sqrt{k_z}\cos z\sqrt{a}}\frac{1}{\sqrt{t}} \times \int_0^\infty \frac{e^{-u^2}\left[\cos\frac{z-h}{\sqrt{t}}\sqrt{4\alpha k_z u^2+\beta t}+\cos\frac{z+h}{\sqrt{t}}\sqrt{4\alpha k_z u^2+\beta t}\right]}{1+\frac{At}{4k_z u^2}\sin\frac{H}{\sqrt{t}}\sqrt{4k_z\alpha u^2+\beta t}}\,du.$$

In the limit, for $t = 0$, we shall have

$$q_{z1} = \lim_{t\to 0}\frac{\cos h\sqrt{a}}{2\sqrt{\pi k_z t}\cos z\sqrt{a}}\left\{\exp\left[-\frac{(z-h)^2}{4k_z t}\right]+\exp\left[-\frac{(z+h)^2}{4k_z t}\right]\right\},$$

i.e., for $t = 0$, $q_{z1} = \delta(z-h)$, $h \le H$.

Thus, we have established that expression (10) satisfies equation (4) in the layer $0 \le z \le H$, as well as the boundary and initial conditions of the problem. Let us note that, when $u_z = 0$, and hence also $a = 0$, solution (10) passes into the known solution of the equation

$$\frac{\partial q_z}{\partial t} = k_z\frac{\partial^2 q_z}{\partial z^2}$$

with the boundary condition

$$\lim_{z\to 0}\left(k_z\frac{\partial q_z}{\partial z}\right) = 0,$$

$$q_z = \frac{1}{2\sqrt{\pi k_z t}}\left\{\exp\left[-\frac{(z-h)^2}{4k_z t}\right]+\exp\left[-\frac{(z+h)^2}{4k_z t}\right]\right\}.$$

An expression analogous to (12) can also be obtained for the layer $H \le z \le \infty$. Of greatest interest in questions of impurity propagation is the calculation of the concentration at the earth's surface $z = 0$. Setting $z = 0$ in (10) and making the substitution $\sqrt{\rho} = u$, we obtain

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Fig. 2

Figure 2: Fig. 2

$$q_{z1} = \frac{\exp\left[-\frac{u_z^2}{4k_z}\,t\right]\cos h\sqrt{a}}{\pi k_z}\int_0^{\infty}\frac{\exp\left[-\frac{u^2}{4k_z}\,t\right]\cos h\sqrt{\alpha u^2+\beta}}{1+\frac{A}{u^2}\sin^2 H\sqrt{\alpha u^2+\beta}}\,du. \qquad (11)$$

To compute integral (11), we split it into two parts

$$J = J_1 + J_2 = \int_0^{u_0}\frac{\exp\left[-\frac{u^2}{4k_z}t\right]\cos h\sqrt{\alpha u^2+\beta}}{1+\frac{A}{u^2}\sin^2 H\sqrt{\alpha u^2+\beta}}\,du + \int_{u_0}^{\infty}\frac{\exp\left[-\frac{u^2}{4k_z}t\right]\cos h\sqrt{\alpha u^2+\beta}}{1+\frac{A}{u^2}\sin^2 H\sqrt{\alpha u^2+\beta}}\,du.$$

Let u_0 be such that for $u \geq u_0$

$$J_2 \simeq \int_{u_0}^{\infty}\exp\left[-\frac{u^2}{4k_z}t\right]\cos h\sqrt{\alpha u^2+\beta}\,du;$$

then

$$\max J_2 < \int_{u_0}^{\infty}\exp\left[-\frac{u^2}{4k_z}t\right]\,du = \sqrt{\frac{\pi k_z}{t}}\left[1-\operatorname{erf}\frac{u_0\sqrt{t}}{2\sqrt{k_z}}\right]. \qquad (A)$$

Thus, from inequality (A) one can determine such a u_0 that $\max J_2 \ll J_1$.

In the integrand, the quantity H enters mainly under the sine sign. Consequently, the quantity H can affect only the choice of the value u_0, while the value of the integral J will depend only weakly on H; indeed, the calculations confirm this (see Fig. 2).

Fig. 2

It follows from this that the profile $u_z(z)$ chosen by us for $H \leq z < \infty$ does not substantially affect the distribution of concentration near the earth' s surface ($z = 0$). The function $u_z(z)$ chosen by us generally coincides with the u_z observed in nature on the interval $0 \leq z \leq H$.

By formula (11) two examples were calculated for $h = 0$:

1. $H = 1000$ m; $u_z = 1$ m/sec; $k_z = 10$ m^2/sec.
2. $H = 500$ m; $u_z = 0.5$ m/sec; $k_z = 10$ m^2/sec.

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The calculation was also carried out by the formula obtained from the solution of the turbulent diffusion equation for the case $u_z = \text{const}$ for $0 \leq z \leq \infty$, with the boundary condition

$$(k_z \partial q_z / \partial z - u_z q_z)_{z=0} = 0;$$

it was assumed here that $z = h = 0$, $u_z = \bar{u}_z = 0.5$ and 1 m/sec.

The results of the calculation, shown in Fig. 2, indicate:

1) Even in the best case, when $\bar{u}_z = 1$ m/sec, averaging the vertical component of the wind velocity in the turbulent diffusion equation can lead to a difference in the calculated concentrations by a factor of 2–3 compared with the concentration calculated using the wind profile $u_z(z)$, which is close to the actual one.

2) The distribution of concentration near the earth' s surface is not substantially affected by the form of the function $u_z(z)$ in the region $H \leq z < \infty$.

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Note: Figure translations are in progress. See original paper for figures.

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