THEORY OF ELASTICITY

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Abstract

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THE PLANE PROBLEM OF THE THEORY OF ELASTICITY FOR A WEDGE WITH GIVEN STRESSES OR DISPLACEMENTS ON THE BOUNDARY

(Presented by Academician S. L. Sobolev, 19 XI 1959)

We consider the problem of finding two functions $\varphi(z)$ and $\psi(z)$, regular inside the wedge $-\alpha\pi/2 < \arg z < \alpha\pi/2$, and satisfying the boundary condition

$$\varkappa\varphi(z_1) - z_1\overline{\varphi'(z_1)} - \overline{\psi(z_1)} = -f(z_1). \tag{1}$$

Here $\arg z_1 = \pm\alpha\pi/2, \quad 0 < |z_1| < \infty, \quad 0 < \alpha < 2;\ f(z_1)$ is a given function.

The constant $\varkappa = -1$ in the case of the first fundamental problem and $\varkappa > 1$ for the second fundamental problem of the theory of elasticity (in the terminology of N. I. Muskhelishvili (^{1})).

Map the wedge onto the right half-plane $\operatorname{Re} s > 0$ by the function $z = s^\alpha$, and denote

$$\Phi(s) = \varphi(s^\alpha), \qquad \Psi(s) = \psi(s^\alpha) + s\,\frac{e^{-i\pi\alpha}}{\alpha}\,\Phi'(s),$$

$$m = \frac{\sin\pi\alpha}{\pi\alpha}, \qquad s = \sigma + i\tau, \qquad f_1(\tau) + if_2(\tau) = f(\tau^\alpha e^{i\pi\alpha/2}). \tag{2}$$

The functions $\Phi(s)$ and $\Psi(s)$, regular in the right half-plane, satisfy on its boundary the condition

$$\varkappa\Phi(i\tau) - \delta(\tau)\,2\pi m\tau\overline{\Phi'(i\tau)} - \overline{\Psi(i\tau)} = -f_1(\tau) - if_2(\tau), \tag{3}$$

where $\delta(\tau) = 1$ for $\tau > 0$; $\delta(\tau) = 0$ for $\tau < 0$.

Assume $\Phi(\infty) = 0$. By Harnack' s theorem, equality (3) is equivalent to two functional equations

$$\varkappa\Phi(s) + m\int_0^\infty \frac{\tau\,\overline{\Phi'(i\tau)}}{i\tau - s}\,d\tau = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f_1(\tau) + if_2(\tau)}{i\tau - s}\,d\tau \equiv A(s); \tag{4}$$

$$\Psi(s) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{f_1(\tau) - if_2(\tau)}{s - i\tau}\,d\tau - m\int_0^\infty \frac{\tau\Phi'(i\tau)}{s - i\tau}\,d\tau. \tag{5}$$

Thus, the problem reduces to solving the functional equation (4) for the function $\Phi(s)$, after which $\Psi(s)$ is computed by formula (5).

Applying to equation (4) the one-sided Laplace transform

$$\Phi(s) = \int_0^\infty u(x)e^{-sx}\,dx, \qquad A(s) = \int_0^\infty F(x)e^{-sx}\,dx, \tag{6}$$

after known transformations (2) we obtain the integral equation for $u(x)$:

$$\varkappa u(x) - m\int_0^\infty \frac{y\,u(y)}{(x+y)^2}\,dy = F(x). \tag{7}$$

The kernel of equation (7) is homogeneous of degree −1; therefore the solution is obtained in closed form by means of the Riemann-Mellin integral transform.

$$\varkappa u(x) = F(x) + \int_0^\infty tF(tx)M(t)\,dt + \int_0^\infty t\overline{F(tx)}N(t)\,dt. \tag{8}$$

Here

$$M(t) = \frac{m^2}{\pi^2\varkappa^2 t}\int_0^\infty \frac{\cos\left(\frac{x}{\pi}\ln t\right)}{\left(\frac{\operatorname{sh} x}{x}\right)^2 - \frac{m^2}{x^2}}\,dx,$$

$$N(t) = \frac{m}{\pi^2\varkappa t}\int_0^\infty \frac{\frac{\operatorname{sh} x}{x}\cos\left(\frac{x}{\pi}\ln t\right)}{\left(\frac{\operatorname{sh} x}{x}\right)^2 - \frac{m^2}{x^2}}\,dx. \tag{9}$$

By direct integration of formulas (8) and (6) we obtain the following expression for $\Phi(s)$:

$$\varkappa\Phi(s) = A(s) + \int_0^1 \left[A\left(\frac{s}{t}\right) + A(st)\right]M(t)\,dt + \int_0^1 \left[\overline{A\left(\frac{s}{t}\right)} + \overline{A(st)}\right]N(t)\,dt. \tag{10}$$

Table 1 gives numerical values of the functions $M(t)$ and $N(t)$ for various values of the parameter $m/\varkappa$. These functions are not expressible in elementary form and, for $t = 0$, have a singularity of the form

$$M(t) = Rt^{\beta-1} + M_0(t),$$

$$\operatorname{sign}\left(\frac{m}{\varkappa}\right) N(t) = +Rt^{\beta-1} + N_0(t). \tag{11}$$

$M_0(t)$ and $N_0(t)$ are continuous functions; β is the smallest positive root of the equation

$$\frac{\sin \pi\beta}{\pi\beta} = \left|\frac{m}{\varkappa}\right| \quad (0 < \beta < 1); \qquad R = \frac{|m|\beta}{2(|m| - |\varkappa| \cos \pi\beta)}.$$

Table 1

t		$m/\varkappa =$ 0.1	$m/\varkappa =$ 0.4	$m/\varkappa =$ $2/\pi$	$m/\varkappa =$ 0.9
0.1	M	0.0101	0.1878	0.6396	3.0971
0.1	N	0.0836	0.4027	0.9150	3.4099
0.4	M	0.0039	0.0694	0.2237	0.9462
0.4	N	0.0513	0.2272	0.4460	1.2252
0.7	M	0.0024	0.0424	0.1354	0.5601
0.7	N	0.0348	0.1522	0.2921	0.7594
1.0	M	0.0016	0.0300	0.0958	0.3946
1.0	N	0.0251	0.1098	0.2097	0.5399
1.0	b_1	−0.0281	−0.0725	−0.0882	−0.0966

For practical computations by formula (10), $M_0(t)$ and $N_0(t)$ should be approximated by simple piecewise-analytic functions. For qualitative investigations, with an accuracy in determining the stresses of the order of 5-10%, one may take the approximation

$$M_0(t) \approx a_0 = \frac{|m|}{2}\left(\frac{|m|}{\varkappa^2 - m^2} - \frac{1}{|m| - |\varkappa| \cos \pi\beta}\right), \tag{12}$$

$$N_0(t) \approx b_0 + b_1(2t - 1),$$

where

$$b_0 = \frac{|m|}{2}\left(\frac{|\varkappa|}{\varkappa^2 - m^2} - \frac{1}{|m| - |\varkappa| \cos \pi\beta}\right),$$

$$b_1 = \frac{6|m|}{|\varkappa|}\int_0^\infty \frac{\operatorname{sh} x/x\,dx}{[(\operatorname{sh} x/x)^2 - m^2/x^2]\,(\pi^2+x^2)} + \frac{3|m|\beta}{(|m|-|\varkappa|\cos\pi\beta)(\beta+1)} - 3b_0.$$

Numerical values of b_1 are given in Table 1.

Let in what follows $\alpha < 1$. As a concrete example of calculations by the method set forth, we shall investigate the distribution of stresses in a wedge under the action of a concentrated force P, applied to one of the

its faces at a distance r_0 from the vertex. Let the angle formed by the direction of the force with the outward normal be denoted by γ. In the present case $x = -1$, $\beta = \alpha$;

$$f(z_1) = \begin{cases} -Pe^{i(\pi\alpha/2+\gamma)}, & \text{for } \operatorname{Im} z_1 < r_0 \sin\dfrac{\pi\alpha}{2}, \\ 0, & \text{for } \operatorname{Im} z_1 > r_0 \sin\dfrac{\pi\alpha}{2}, \end{cases} \tag{13}$$

$$A(s) = \frac{P}{2\pi i} e^{i(\gamma+\pi\alpha/2)} \ln(s-s_0), \qquad \text{where } s_0 = i r_0^{1/\alpha} = i\tau_0.$$

From formulas (10) and (13) we find

$$\begin{aligned} \frac{2\pi\alpha r_0}{P}\, i e^{-i\gamma}\varphi'(z) = \zeta^{1-\alpha}\Bigg\{ & \frac{1}{\zeta-1} + \int_0^1 M_0(t)\left[\frac{1}{\zeta-t} + \frac{t}{t\zeta-1}\right] dt \\ & - e^{-i(\pi\alpha+2\gamma)} \int_0^1 N_0(t)\left[\frac{1}{\zeta+t} + \frac{t}{t\zeta+1}\right] dt \Bigg\} \\ & + \zeta^{1-\alpha} R \Bigg\{ I(\zeta,\alpha) - \frac{1}{\zeta} I\left(\frac{1}{\zeta}, 1+\alpha\right) \\ & + e^{-i(\pi\alpha+2\gamma)}\left[I(-\zeta,\alpha) + \frac{1}{\zeta} I\left(-\frac{1}{\zeta}, 1+\alpha\right)\right]\Bigg\}. \end{aligned} \tag{14}$$

Here

$$\zeta = \frac{s}{s_0} = \left(\frac{r}{r_0}\right)^{1/\alpha} e^{\frac{i}{\alpha}(\arg z - \frac{\pi\alpha}{2})}, \qquad I(\zeta,\alpha) = \int_0^1 \frac{t^{\alpha-1}}{\zeta - t}\, dt.$$

For the approximate computation of $I(\zeta,\alpha)$, the cases $|\zeta| < 1$ and $|\zeta| > 1$ should be separated.

For $|\zeta| < 1$,

$$I(\zeta,\alpha)=\int_1^\infty \frac{t^{\alpha-1}\,dt}{\zeta-t}-\frac{e^{i\pi\alpha}\pi}{\sin\pi\alpha}\,\zeta^{\alpha-1}=\zeta\int_0^1 \frac{t^{1-\alpha}\,dt}{t\zeta-1}-\frac{e^{i\pi\alpha}\pi}{\sin\pi\alpha}\,\zeta^{\alpha-1}-\frac{1}{1-\alpha}.$$

For $|\zeta|>1$,

$$I(\zeta,\alpha)=\frac{1}{\alpha\zeta}+\frac{1}{\zeta}\int_0^1 \frac{t^\alpha}{\zeta-t}\,dt.$$

In the last two integrals one may use the approximate formula (3)

$$t^\alpha \simeq t\,\frac{(2-\alpha)+\alpha t}{\alpha+(2-\alpha)t}.$$

Let us compute the normal stresses σ_r on the faces of the wedge in the case $\gamma=0$:

$$\sigma_r=4\operatorname{Re}\varphi'(z)=\frac{2P}{\pi\alpha r_0}\,\beta(\zeta). \qquad (15)$$

For $\zeta>0$,

$$\beta(\zeta)=\zeta^{1-\alpha}\left\{\pi M(\sigma)+\sin\pi\alpha\int_0^1 N(t)\left[\frac{1}{t+\zeta}+\frac{t}{t\zeta+1}\right]dt\right\}. \qquad (16)$$

For $\zeta=-x<0$,

$$\beta(-x)=x^{1-\alpha}\sin\pi\alpha\left\{\frac{1}{1+x}+\int_0^1 M(t)\left[\frac{1}{t+x}+\frac{t}{tx+1}\right]dt+\pi x^{1-\alpha}N(x)\right\}.$$

Table 2 gives the values of $\beta(\zeta)$ and $\beta(-x)$ for a right-angle wedge, when $\alpha=1/2$. In compiling this table, the above approximation of the functions $M_0(t)$ and $N_0(t)$ was used. To estimate the error of the table, we note that for $\zeta=x=1$ the integrals (16) are easily computed exactly: $\beta(1)=-0.234$; $\beta(-1)=0.183$.

We note that in paper (4) an erroneous solution of this problem is given. Passing in formula (14) to the limit as $r_0\to 0$, we obtain Michell' s well-known solution (5) of the problem of the deformation of a wedge by a force applied at its vertex:

$$e^{-i(\pi\alpha/2+\gamma)}\frac{2\pi\alpha}{P}\,z\varphi'(z)=1+2\int_0^1 M(t)\,dt-2e^{-i(\pi\alpha+2\gamma)}\int_0^1 N(t)\,dt=$$

$$= \frac{1}{1-m^2} + e^{-i(\pi\alpha+2\gamma)}\frac{m}{1-m^2}.$$

With the above-indicated approximation of the functions $M_0(t)$ and $N_0(t)$, in the case $\alpha = \frac{1}{2}$ the following approximate solution of Michell' s problem is obtained:

$$\frac{\pi}{P}e^{i(\pi/4-\gamma)}z\varphi'(z) \approx 1.6814 - 1.0705e^{2i(\pi/4-\gamma)}.$$

The exact value of the right-hand side is: $1.6815 - 1.0705e^{2i(\pi/4-\gamma)}$. For the solution of the problem in the case of an arbitrary load on the upper face of the wedge, knowing the function $\beta(\zeta)$, one can compute $4\,\mathrm{Re}\,\varphi'(z)$ by integrating the right-hand side of expression (15) with respect to r_0 over the interval $(0,\infty)$. In particular, for $\tau(r_0) \equiv 0$, $\rho(r_0) = -p = \text{const}$, in this way we obtain (with the aid of the indicated approximation of $M_0(t)$ and $N_0(t)$) the following approximate solution:

$$4\varphi'(z) \approx -1.011\,p = \text{const}.$$

Table 2

ζ, x	$\beta(\zeta)$	$\beta(-x)$	ζ, x	$\beta(\zeta)$	$\beta(-x)$
0	0.0000	0	1.3	−0.2375	0.2311
0.1	−0.1027	−0.0655	1.4	−0.2360	0.2479
0.2	−0.1301	−0.0539	1.5	−0.2355	0.2629
0.3	−0.1483	−0.0373	1.6	−0.2353	0.2746
0.4	−0.1646	−0.0165	1.7	−0.2352	0.2872
0.5	−0.1801	+0.0106	1.8	−0.2350	0.2997
0.6	−0.1949	0.0385	1.9	−0.2350	0.3099
0.7	−0.2091	0.0796	2.0	−0.2350	0.3199
0.8	−0.2232	0.1219	2.1	−0.2349	0.3290
0.9	−0.2369	0.1694	2.2	−0.2348	0.3372
1.0	−0.2503	0.2074	5.0	−0.2265	0.3990
1.1	−0.2438	0.2183	10.0	−0.2025	0.3601
1.2	−0.2398	0.2243			

The exact solution of the last problem, found by M. Lévy, is determined by the formula

$$4\varphi' z = -p.$$

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CITED LITERATURE

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Note: Figure translations are in progress. See original paper for figures.

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