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Abstract

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N. N. MEIMAN

ON THE THEORY OF POLYNOMIALS LEAST DEVIATING FROM ZERO

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1°. In the present paper a certain general construction is described which makes it possible to find the form of the solution of all extremal problems having a unique solution with a prescribed number of points of maximum deviation from zero. Such are, for example, the well-known problems on the polynomial of least deviation with several prescribed leading coefficients, or on the polynomial least deviating from zero on several prescribed intervals. We confine ourselves to describing the construction for polynomials, but with the corresponding changes the construction carries over to entire functions and other classes of functions, which makes it possible to solve in these classes a number of extremal problems.

2°. Let $Q_n(z) = U_n(x,y) + iV_n(x,y)$ be an arbitrary real polynomial of the form $z^n + c_1 z^{n-1} + \cdots$, and let $A > 0$. By $\mu(Q_n; A)$ we denote the set of points at which $V_n(x,y) = 0$ and $|Q_n(z)| \le A$. The set $\mu(Q_n; A)$ is closed, bounded, and symmetric with respect to the real axis. The complement of $\mu(Q_n; A)$ is connected, i.e. is a domain. We shall denote this domain by $\mathfrak{G}(Q_n; A)$. Each of the branches of the function $(A^2 - Q_n^2(z))^{1/2}$ is single-valued in $\mathfrak{G}$. Choose the branch whose value tends to $iQ_n(z)$ as $z \to \infty$, and define the function

$$\omega(z; Q_n; A) \equiv A^{-1}\left(Q_n(z) - i\sqrt{A^2 - Q_n^2(z)}\right).$$

The function $\omega(z)$ is admissible in $\mathfrak{G}$ (see [1]) and

$$\overline{\omega}(z) = A^{-1}\left(Q_n(z) + i\sqrt{A^2 - Q_n^2(z)}\right).$$

Since $\omega(z)\overline{\omega}(z) = 1$, the functions $\omega(z)$ and $\overline{\omega}(z)$ have no finite zeros. The function $\zeta(z) = \overline{\omega}(z)/\omega(z)$ has at the point $z = \infty$ a zero of order $2n$; therefore the set M (see [1]) is empty and $\omega(z) \in HB(\mathfrak{G})$. Introduce the function

$$\varphi(z) = -i2^{-1}\ln\zeta(z) = i\ln\omega(z).$$

The function $\varphi(z)$ is real on μ and strictly increases along each component of μ. At the points of $\mathfrak{G}$, $\operatorname{Im}\varphi(z) > 0$. The index $S_0 = 0$ (see [1]), and each root of multiplicity k of the equation

$$\sqrt{A^2 - Q_n^2(z)} = 0$$

has in $\mathfrak{G}$ an angular neighborhood of aperture $k^{-1}\pi$ (see Theorem 3 of ([1])); in particular, all boundary points of the set μ are roots of multiplicity 1/2. There cannot exist a closed contour along which $\Delta \arg \omega(z) \neq 0$. It follows from this that a component of the set μ can contain no more than one real segment. Suppose that μ has p such components of the first kind $\mu_1, \dots, \mu_p$. In addition, μ may contain $2l$ components $\gamma_1, \overline{\gamma}_1, \dots, \gamma_l, \overline{\gamma}_l$ of the second kind, containing no real segments. We shall consider only polynomials $Q_n(z)$ in general position, i.e. such that in the coefficient space the corresponding points do not lie on a certain finite number of algebraic surfaces. The exceptional surfaces are determined from the course of the argument. Each component μ_j of the first kind consists of a real base and, possibly, a certain number of analytic arcs $\delta^\nu + \overline{\delta}^\nu$ orthogonal to the base. On δ^ν the only zero of the equation

$$\sqrt{A^2 - Q_n^2(z)} = 0$$

is served by

end of an arc δ^ν. At a real point δ^ν, x_ν, $Q_n'(x_\nu) = 0$. Let q be the total number of such arcs $\delta^\nu + \bar{\delta}^\nu$. Each component γ is a smooth simple arc, and $\sqrt{A^2 - Q_n^2(z)}$ vanishes only at its endpoints. From counting the number of zeros of $\sqrt{A^2 - Q_n^2(z)}$ along μ there follows the relation $p + q + 2l = n$. The number of real zeros of the derivative $Q_n'(z)$ is equal to $q + (p - 1 + 2m)$, q of these zeros lying on μ. The number of complex zeros of $Q_n'(z)$ is equal to

$$n - 1 - (q + p - 1 + 2m) = 2(l - m).$$

Let $\mathfrak{G}_+(Q_n; A)$ be the intersection of the domain $\mathfrak{G}(Q_n; A)$ with the half-plane $\operatorname{Im} z \geq 0$. The boundary $\mathfrak{G}_+$ consists of p real bases of components of the first kind, $p + 1$ intervals of the real axis supplementary to them, q arcs δ^ν, and components $\gamma_1, \dots, \gamma_l$. Through each complex zero z_0 $(\operatorname{Im} z_0 > 0)$ of the derivative $Q_n'(z)$ pass two orthogonal curves $\arg \omega(z) = \arg \omega(z_0)$. For a polynomial $Q_n(z)$ in general position we may assume that these curves do not pass through other zeros of $Q_n'(z)$. Along one of these curves $|\omega(z)|$ strictly decreases on both sides from the point z_0, and along the other curve $|\omega(z)|$ strictly increases. The first of these curves connects two different components μ (only one of them may be a component of the first kind). Let ξ_0 be a real zero of $Q_n'(z)$, $\xi_0 \bar{\in} \mu$, and let $|Q_n(x)|$ have a minimum at the point ξ_0 (x real). Along the curve $\arg \omega(z) = \arg \omega(\xi_0)$, $|\omega(z)|$ strictly decreases; therefore this curve ends at one of the endpoints of the component γ.

Thus, in $\mathfrak{G}_+$ there exist: 1) $l - m$ curves, each of which joins either two different components of the second kind, or a component of the first kind with a component of the second kind; along each of these curves $\arg \omega(z) = \text{const} \neq 0 \pmod{\pi}$; 2) m curves, each of which joins the real axis with a component of the second kind; along these curves $\arg \omega(z) \equiv 0 \pmod{\pi}$. Cuts made along all these curves do not divide $\mathfrak{G}_+$ into disconnected parts, since in that case there would exist a closed curve along which $\Delta \arg \omega(z) < 0$. The simply connected domain obtained from $\mathfrak{G}_+$ by means of these cuts will be called the fundamental domain $F(Q_n; A)$, and its boundary will be denoted by $\dot{F}$.

3°. The function $t = \varphi(z)$ conformally maps the domain $F(Q_n; A)$ onto the domain $T(Q_n; A)$; we normalize $\varphi(z)$ by the condition $\operatorname{Re}\varphi(-\infty) = -n\pi$. Traversal along $\dot{F}$, beginning with $x = -\infty$, determines the numbering of all elements of the boundary $\dot{F}$. The function $\varphi(z)$ can be represented in the form

$$\varphi(z) = -\int_{z_\nu}^{z} \frac{Q_n'(z)\,dz}{\sqrt{A^2 - Q_n^2(z)}} - (n - \nu + 1)\pi, \qquad z \in F, \tag{1}$$

where z_ν is the ν-th zero of the function $\sqrt{A^2 - Q_n^2(z)}$ along $\dot{F}$, and the path of integration lies in F.

Denote the real zeros of $Q_n'(z)$ lying at the bases of the arcs $\delta_1, \dots, \delta^q$ by $x_1, \dots, x_q$, and the real zeros lying on the supplementary intervals by $\xi_1^k, \xi_2^k, \dots$, where $k = 0, 1, \dots, p$ is the number of the supplementary interval. The points ξ_{2i+1}^0 and ξ_{2i}^k, $k \geq 1$, are points of minimum for $|Q_n(x)|$, and through them pass the cuts C_i^k, $k = 0, 1, \dots, p$. The zeros of $Q_n'(z)$ in the upper half-plane will be denoted by $\zeta_1, \dots, \zeta_{l-m}$, and the cuts passing through them by B_i. Let ζ be a current point of $\dot{F}$, and let $N(\zeta)$ be the number of zeros of the function $\sqrt{A^2 - Q_n^2(z)}$ on the arc $\dot{F}(-\infty, \zeta]$; then

$$\arg\omega(\zeta) - \arg\omega(-\infty) = -(N - 1 + \theta)\pi,$$

where $0 \leq \theta < 1$, $N \geq 1$. The corresponding points of the two banks of a cut will be denoted by ζ_- and ζ_+; each zero of the polynomial $Q_n(z)$ lying on the cut δ^ν or γ is counted twice.

Consider the images of the elements $\dot{F}$ under the mapping $t = \varphi(z)$.

1) Each component γ is mapped onto one or several segments of the real axis of total length 2π. The points $\varphi(\zeta_-)$ and $\varphi(\zeta_+)$ are situated symmetrically with respect to the point $(N(\zeta_-) + j - n)\pi$, where j is the number of components of the second kind on the arc $\dot{F}(\zeta_-, \zeta_+)$.

2) The negative shore of the cut B_i passes into both shores of the following cut in the t-plane: $0 \leq \operatorname{Im} t \leq \ln|\omega(\zeta_i)|$, $\operatorname{Re} t = -[N(\zeta_{i,-}) + \theta_i - 1 - n]\pi$, where $0 < \theta_i < 1$; the image of the positive shore of the cut B_i is shifted parallel by $2m_i\pi$, where m_i is the number of components of the second kind on the path joining both shores of the cut.

3) The negative shore of the arc δ^ν is mapped onto one or several real intervals of total length $\tau^\nu\pi$, $0 < \tau^\nu < 1$, separated from one another by intervals of length a multiple of 2π. The images of the points ζ_- and ζ_+ of the arc δ^ν are situated symmetrically with respect to the point $[N(\zeta_-) + j - n]\pi$, where j is the number of components γ on the portion of the path $\dot{F}(\zeta_-, \zeta_+)$.

4) The negative shore of the cut C_i^k, $k \geq 1$, passes into a portion of the positive shore of the cut $\operatorname{Re} t = [N(\xi_{2i-1}^k) - 1 - n]\pi$, $\ln|\omega(\xi_{2i}^k)| \geq \operatorname{Im} t \geq 0$;

the length of the whole cut is equal to $\ln|\omega(\xi^k_{2i-1})|$; for $k=0$ the indices $2i$ and $2i-1$ must be increased by 1. By the point ξ^0_0 is meant $x=-\infty$. The positive shore of the cut $C^k_i,\ k\geq 1$, is mapped onto a portion of the negative shore of the cut $\operatorname{Re} t=[N(\xi^k_{2i+1})-1-n]\pi,\ 0\leq \operatorname{Im} t\leq \ln|\omega(\xi^k_{2i})|$; the length of the whole cut is equal to $\ln|\omega(\xi^k_{2i+1})|$; for $k=0$ the indices $2i$ and $2i+1$ must be increased by 1. If ξ^p_{2i} is the greatest real zero of $Q'_n(z)$, then by the point ξ^p_{2i+1} one must understand $x=+\infty$.

4°. $T(Q_n;A)$ is an infinite half-strip whose base is the real interval $[-n\pi,0]$. On the left and on the right T is bounded by the half-lines $\operatorname{Re} t=-n\pi,\ \operatorname{Re} t=0,\ \operatorname{Im} t\geq 0$. In T there are $l-m$ pairs of cuts parallel to the imaginary axis, of length $\ln|\omega(\zeta_i)|,\ i=1,\dots,l-m$, and with bases at the points $\operatorname{Re} t=-\arg\omega(\zeta_{i,-})$, $\operatorname{Re} t=-\arg\omega(\zeta_{i,+})$, and $p-1+m$ cuts of length $\ln|\omega(\xi^0_{2i})|$ for $k=0$ and $\ln|\omega(\xi^k_{2i+1})|$ for $k\geq 1$. The bases of these cuts lie at the points $-\arg\omega(\xi^0_{2i})=[N(\xi^0_{2i})-1-n]\pi$ and $-\arg\omega(\xi^k_{2i+1})=[N(\xi^k_{2i+1})-1-n]\pi$. For T, considered as the image of F, besides the indicated $l-m$ continuous parameters θ_i and $l-m+p-1+m$ continuous parameters $\ln|\omega(\zeta_i)|,\ \ln|\omega(\xi^0_{2i})|$ and $\ln|\omega(\xi^k_{2i+1})|,\ k\geq 1$, and the integer parameters $N(\zeta_{i,\mp})$, $N(\xi^k_{2i})$, $N(\xi^k_{2i+1})$, the following quantities are of essential significance: 1) the lengths of the images of the cuts C^k_i, equal to $\ln|\omega(\xi^0_{2i+1})|$ and $\ln|\omega(\xi^k_{2i})|,\ k\geq 1$; 2) the parameters $\tau^\nu,\ \nu=1,\dots,q$. The total number of continuous parameters of T is equal to $(l-m)+(l+p-1+q)=n-1$.

The number of distinct combinations of the integer parameters determining the type of the half-strip T is equal to

$$\sum_{p,q,l,m}\binom{p+q-1}{q}\binom{p+m}{m}R_{l,m,p+q},\qquad \text{where } 0\leq p,\ 0\leq q,\ 0\leq m\leq l; \quad (2)$$

$$R_{l,m,\nu}=\sum_{i=m}^{l}\binom{i-1}{m-1}2^{l-i}\left[1\cdot\binom{\nu}{1}2^{-1}+\binom{l-i-1}{1}\binom{\nu}{2}2^{-2}+\cdots\right.$$
$$\left.\cdots+\binom{l-i-1}{\nu}\binom{\nu}{\nu}2^{-\nu}\right]. \quad (3)$$

As always, $p+q+2l=n$. When $l-i\leq\nu$, the summation in the bracket terminates at the $(l-i)$-th term. When $i=l$, the entire bracket is replaced by unity.

$$R_{0,0,\nu}=1,\qquad R_{1,0,\nu}=\nu,\qquad R_{1,1,\nu}=1. \quad (4)$$

It can be shown that to each half-plane T with some set of continuous and integer-valued parameters there corresponds a polynomial $Q_n(z)$, a constant A, and a domain $F(Q_n;A)$.

Theorem. Two polynomials $Q_n^1(z)$, $Q_n^2(z)$ and quantities A_1, A_2 correspond to a half-plane T with one and the same parameters if and only if these polynomials and quantities are connected by the relations

$$Q_n^2(z) = a^{-n} Q_n^1(az+b), \qquad A_2 = a^{-n} A_1, \qquad a > 0. \tag{5}$$

Proof. The function $\varphi_1^{-1}[\varphi_2(z)]$ conformally maps the fundamental domain $F(Q_n^2; A_2)$ onto the fundamental domain $F(Q_n^1; A_1)$. In this case the points ζ_-^2 and ζ_+^2 of the boundary $\dot{F}(Q_n^2; A_2)$ pass into the points ζ_-^1 and ζ_+^1 of the boundary $\dot{F}(Q_n^1; A_1)$, and all the cuts in $F(Q_n^1; A_1)$ and $F(Q_n^2; A_2)$ can be glued together; i.e., $\varphi_1^{-1}[\varphi_2(z)]$ conformally maps the upper half-plane onto itself and, consequently, $\varphi_1^{-1}[\varphi_2(z)] = az + b$, or $\varphi_2(z) = \varphi_1(az+b)$.

Institute of Theoretical and Experimental Physics
Academy of Sciences of the USSR

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[1] N. N. Meiman, DAN, **124**, No. 6 (1959). [2] N. N. Meiman, DAN, **125**, No. 5 (1959)*.

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* **Correction to paper** (2)**.** Theorem 4 and the lower estimate in inequality (2) are proved only under the assumption that zero is not an asymptotic value in $\mathfrak{R}$ of the function $u[z(\zeta)] = f(\zeta) : v[z(\zeta)]$. Otherwise, in Theorem 4 the word "right" must be replaced by "left." If $f(\zeta)$ is an entire function, then the number of nontrivial zeros of $u(\zeta) - f(\zeta)$ in the ζ-plane is equal to, or no greater than, $a^0 + \delta_- + \delta_+$. In the case $0 < u(x_i) : f(x_i) < 1$, $2n_i = 1$ in inequality (2). In the example, a^0 must be doubled, and for $0 < \cos\alpha : f(0) < 1$ the term 2 in relation (3) must be replaced by 1.

Note: Figure translations are in progress. See original paper for figures.