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Abstract

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DIFFRACTION OF SOUND BY A PERIODICALLY UNEVEN AND INHOMOGENEOUS SURFACE

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We consider the plane problem of the diffraction of sound by a periodically uneven and inhomogeneous surface with a given normal acoustic conductivity. Starting from the integral equation for the acoustic pressure p on the surface ([1]), which is a special case of Green' s formula, and retaining the notation of ([1]), we obtain for the Fourier transform of p the finite-difference equation

$$F(u)=2\Pi(u)+\sum_{m=-\infty}^{+\infty}\alpha_m(u)F(u+mq), \tag{1}$$

where

$$\alpha_m(u)=\frac{k}{2d}\int_0^d e^{-imqx}\,dx\int_{-\infty}^{+\infty}e^{iu\tau}\left\{\frac{H_1^{(1)}(kr)}{r}\,i[\zeta(x+\tau)-\zeta(x)-\tau\zeta'(x)]+\right.$$

$$\left.+H_0^{(1)}(kr)\frac{\eta(x)}{n_z(x)}\right\}d\tau; \tag{2}$$

here $d=2\pi/q$ is the period of the grating;

$$r=\sqrt{\tau^2+[\zeta(x+\tau)-\zeta(x)]^2}.$$

We shall assume further that $\zeta(x)$ is a four-times differentiable function, while $\eta(x)$ is piecewise continuous and has one-sided derivatives at points of discontinuity. We seek the solution of equation (1) in the form

$$F(u)=\sum_{n=-\infty}^{\infty}\frac{2\Pi(u+nq)X_n(u+nq)}{1-\alpha_0(u+nq)}.$$

Hence we obtain an infinite system of linear equations for $X_n(u)$:

$$X_n(u) + \sum_m C_{n,m}(u) X_m(u) = \delta_n, \tag{3}$$

where $\delta_0 = 1,\ \delta_n = 0$ for $n \neq 0$; $C_{n,n}(u) = 0$,

$$C_{n,m}(u) = \alpha_{n-m}(u - nq)/[\alpha_0(u - nq) - 1]$$

for $m \neq n$.

In investigating the coefficients of system (3) it is convenient to separate from the function $\alpha_m(u)$ the singularities contained in it. It can be shown that

$$\alpha_m(u) = \alpha_m^{(1)}(u) + \alpha_m^{(2)}(u),$$

where

$$\alpha_m^{(1)}(u) = i\zeta_m \left[\frac{mqu}{\sqrt{k^2 - u^2}} + \sqrt{k^2 - (u + mq)^2} - \sqrt{k^2 - u^2}\right] +$$

$$+k\eta_m/\sqrt{k^2 - u^2}; \qquad \zeta_m = \frac{1}{d}\int_0^d \zeta(x) e^{-imqx}\, dx;$$

$$\eta_m = \frac{1}{d}\int_0^d [\eta(x)/n_z(x)] e^{-imqx}\, dx; \qquad \mathrm{Re}, \mathrm{Im}\, \sqrt{k^2 - u^2} \geq 0.$$

$\alpha_m^{(2)}(u)$ is expressed by formula (2), if in it $H_1^{(1)}(kr)/r$ and $H_0^{(1)}(kr)$ are replaced, respectively, by the differences

$$[H_1^{(1)}(kr)/r] - [H_1^{(1)}(k|\tau|)/|\tau|]$$

and

$H_0^{(1)}(kr) - H_0^{(1)}(k|\tau|)$. The functions $a_m^{(2)}(u)$ are everywhere continuous and tend to zero both as $u \to \pm\infty$ and as $m \to \pm\infty$. Integrating by parts the corresponding expression for $a_m^{(2)}$ a sufficient number of times and majorizing the integrals thereby obtained, it is not difficult to prove that the Koch condition

$$\sum_{n,m} |C_{n,m}(u)|^2 < \infty$$

is satisfied everywhere, except for the values $u = \pm k + nq$ when $\eta_0 = 0$.* Consequently, (2), for all u (except $u = \pm k + nq$ when $\eta_0 = 0$), the solution of system (3) has the form

$$X_m(u) = \Delta_m(u)/\Delta(u), \tag{4}$$

where $\Delta(u)$ is the determinant of the system,** and $\Delta_m(u)$ is the cofactor of the element $C_{0,m}(u)$. $\Delta(u)$ and $\Delta_m(u)$ can be computed to the desired degree of accuracy by the reduction method (2). It is easy to note the periodicity of the determinant of the system: $\Delta(u + q) = \Delta(u)$.

For $\eta_0 = 0$ and $u \to \pm k + nq$, at least some elements $C_{n,m}(u)$, situated in the n-th row of the determinant $\Delta(u)$, tend to infinity, so that, generally speaking,*** $|\Delta(u)| \to \infty$. At the same time $|\Delta_m(u)| \to \infty$, unless all the elements of the system that go to infinity belong to the central row, which is not included in $\Delta_m(u)$. In this case (4) gives a removable indeterminacy of the type ∞/∞. Indeed, noting that

$$\sqrt{1 - (u/k)^2}\, \alpha_m(u) \to \eta_m \pm imq\zeta_m$$

as $u \to \pm k$, let us multiply the numerator and denominator in (4) by

$$\sqrt{1 - [(u - nq)/k]^2},$$

multiplying by this factor the elements of rows with number n in the determinants $\Delta(u)$ and $\Delta_m(u)$. Then, if $2k \neq Nq$, where N is an integer, the products obtained remain finite also at $u = nq \pm k$. For $2k = Nq$, when an integer number of half-waves fits within the period length, two rows in the determinants go to infinity at once, and to remove the indeterminacy as $u \to nq \pm k$ the determinants should be multiplied by

$$\sqrt{1 - [(u - nq)/k]^2}\sqrt{1 - [(u - nq \mp Nq)/k]^2}.$$

Thus, in all the cases listed, (4) gives a solution of system (3).**** Obviously, $X_m(u) \to \delta_m$ both as $u \to \pm\infty$ and as $m \to \pm\infty$.

The expression for the sound pressure on the reflecting surface will be obtained in the form of the inverse Fourier transform of the function $F(u)$. Thus, for example, in the case of an incident plane wave of the form $p_i = \exp i(k_x^0 X - k_z^0 Z)$,

$$p[x, \zeta(x)] = 2 \sum_{m,n} X_{m-n}(-k_x^n) C_n(k_z^0) \{1/[1 - \alpha_0(-k_x^n)]\} e^{ik_x^m x}, \tag{5}$$

where

$$k_x^n = k_x^0 + nq, \qquad C_n(\chi) = \frac{1}{d}\int_0^d e^{-i[\chi\zeta(x) + nqx]}\, dx.$$

We express the field of sound pressure of the waves reflected from the surface, $p_r(X, Z)$, by Green' s formula through the found values $p(x, \zeta)$. Restricting the consideration to the case $Z \gg \max \zeta(x)$, we obtain

$$p_r(X, Z) = \frac{1}{4\pi} \sum_n \int_{-\infty}^{\infty} e^{i(\varkappa_x X + \varkappa_z Z)} C_n(\varkappa_z) \frac{k}{\varkappa_z} \Phi_n(\varkappa_x)\, d\varkappa_x, \tag{6}$$

* The case where the denominator in the expression for $C_{n,m}(u)$ is equal to zero, which, as can be shown, corresponds to an infinite density of the energy flux flowing out of the reflecting surface when a plane wave is incident on it, is excluded from consideration.

** It can be proved that $\Delta(u) \neq 0$. However, for a surface with sufficiently gentle roughness and small inhomogeneities, when system (3) is completely regular, this is obvious.

*** The exception is the case $\eta_m = \pm imq\zeta_m$ for all m.

**** If the surface structure is such that the coefficients ξ_n, η_n are nonzero only for odd values of n, then $a_{2m}(u) \equiv 0$.

where

$$\Phi_n(-\varkappa_x) = (\varkappa_z/k)F(\varkappa_x^n) + \sum_m [imq\xi_m(\varkappa_x/k) + \eta_m]F(\varkappa_x^{n+m}), \qquad \varkappa_x^n = \varkappa_x + nq,$$

$$\varkappa_z = \sqrt{k^2 - \varkappa_x^2}; \qquad \mathrm{Re}, \mathrm{Im}\, \varkappa_z \geq 0.$$

It is not difficult to obtain the estimate $C_n(\varkappa) = \delta_n - i\varkappa\xi_n + \varepsilon$, $|\varepsilon| \leq \varkappa^2\bar{\xi}^2/2$, where $\bar{\xi}^2$ is the mean-square value of the function $\xi(x)$. It follows from this that $|C_n(\varkappa_x)k/\varkappa_z| \to \infty$ as $\varkappa_x \to \pm k$, $n = 0$. However, $\Phi_0(\varkappa_x)$ then tends to zero and the integrand remains finite. Thus, putting $u = \varkappa_x$ in (1) and grouping the terms containing $k/\varkappa_z$, we have

$$\Phi_0(-\varkappa_x)k/\varkappa_z = 2F(\varkappa_x) - 2\Pi(\varkappa_x) - \sum_m [i\xi_m(\varkappa_z^m - \varkappa_z) + \alpha_m^{(2)}(\varkappa_x)]F(\varkappa_x^m),$$

where

$$\varkappa_z^n = \sqrt{k^2 - (\varkappa_x^n)^2}.$$

In the case of an incident plane wave, (6) reduces to the sum of Bragg spectra (homogeneous and inhomogeneous)

$$p_r(X, Z) = \sum_l e^{i(k_x^l X + k_z^l Z)} \sum_n C_n(k_z^l)\frac{k}{k_z^l} b_{n,l},$$

where

$$b_{n,l}=\sum_s C_s(k_z^0)[1-\alpha_0(-k_x^s)]^{-1}\left[(k_z^l/k)X_{l-n-s}(-k_x^s)+\right.$$

$$\left.+\sum_m(\eta_m-imq\xi_m k_x^l/k)X_{l-n-m-s}(-k_x^s)\right].$$

Resolving the indeterminacy of the type 0/0 contained here, we obtain

$$(k/k_z^l)b_{0,l}=-C_l(k_z^0)+\sum_s C_s(k_z^0)[1-\alpha_0(-k_x^s)]^{-1}\left\{2X_{l-s}(-k_x^s)-\right.$$

$$\left.-\sum_m[i\xi_m(k_z^{l-m}-k_z^l)+\alpha_m^{(2)}(-k_x^l)]X_{l-m-s}(-k_x^s)\right\}.$$

Let us note some consequences of the results obtained:

1) If the incident plane wave propagates along the surface ($k_z^0=0$), then the total field p is everywhere equal to zero. Indeed, the right-hand side of (5) vanishes in the case $k_z^0=0$, since for $\eta_0\neq 0$, $|\alpha_0(\pm k)|=\infty$, while for $\eta_0=0$, as is easy to see from (4), $X_m(\pm k)=0$ for all m.

2) For $\eta_0=0$ and $2k\neq Nq$ we have

$$X_m(mq\pm k)=(\xi_m^\pm+\varepsilon_2)\left[\sum_n \xi_n^\pm C_{-n,0}(\pm k)+\varepsilon_3\right],$$

where $\xi_n^\pm=\eta_n\pm inq\xi_n$; ε_s denotes the sum of products of nondiagonal coefficients of the system (3), consisting of no fewer than s factors. If $M=\max(|\eta_n|,\ |nq\xi_n|)\to 0$, then $\varepsilon_s\to 0$ no slower than M^s. In addition, with accuracy up to terms of order M^2 we have

$$C_{n,0}(\pm k)\simeq-\alpha_n^{(1)}(\pm k-nq)=-\sqrt{k/q}\,\xi_n^\pm/\sqrt{\pm 2n-n^2q/k}.$$

Consequently (at least for sufficiently smooth and homogeneous ($M\ll 1$) surfaces), $|X_m(mq\pm k)|\gg 1$, provided only that $|\xi_m^\pm|$ is not too small compared with $\max|\xi_n^\pm|$. For example, for a rigid surface ($\eta(x)\equiv 0$)

$$X_m(mq\pm k)\simeq$$

$$\simeq\pm im\xi_m'\left\{\sqrt{qk}\sum_{n=1}^{\infty}n^2\xi_n\xi_{-n}\left[(1/\sqrt{\pm 2n-n^2q/k})+(1/\sqrt{\mp 2n-n^2q/k})\right]\right\}^{-1}.$$

On the other hand, if for the incident plane wave we put $k_x^0 = k - mq$, then the spectrum of the m-th order grazing along the surface has amplitude equal to $A_m = 2X_m(mq - k) + O(1)$. Hence it is clear that

the amplitudes of the grazing spectra can attain comparatively large values*. This phenomenon can be observed for any q/k, and it is expressed somewhat less sharply if $Nq = 2k$, but in that case two grazing spectra appear at once. It is also obvious that this phenomenon is characteristic both of an uneven surface and, generally speaking, of a plane periodically inhomogeneous surface.

For $\eta_0 = 0$, $|A_m|$ increases with increasing degree of flatness and homogeneity of the surface and may exceed any prescribed value; for $0 < |\eta_0| \ll 1$, as M decreases, $|A_m|$ first increases, reaching a maximum value (of order $\sqrt{q/(k\eta_0)}$), and then tends to zero.

Obviously, the field of diffracted waves p_r may be interpreted as the result of interference of waves scattered by each cell (with allowance for their diffraction by the remaining cells of the surface). In this case, the contribution of each cell to the resulting field p_r, sufficiently far from the surface, can be characterized by a certain directivity characteristic $\varphi(\theta)$ of the scattered waves; θ is the scattering angle. It can be shown that

$$A_n = \sqrt{1-\varepsilon}\,\varphi(\theta_n)\cos\theta_0/\cos\theta_n,$$

where ε is the relative fraction of energy absorbed by the surface, A_n is the amplitude of the n-th spectrum, $\theta_n = \arcsin(k_x^n/k)$; the normalization condition for $\varphi(\theta)$ is chosen in the form

$$\sum \varphi(\theta_n)\varphi^*(\theta_n)\cos\theta_0/\cos\theta_n = 1$$

(the summation is over all n for which θ_n are real). The factor $\cos\theta_0/\cos\theta_n$ increases without bound as $|\theta_n| \to \pi/2$, while $\varphi(\theta_n) \to 0$, since waves initially scattered at small grazing angles to the surface are scattered by the irregularities and inhomogeneities of the grating into other directions, and the more intensively so, the smaller the grazing angle. In this case the ratio $\varphi(\theta_n)/\cos\theta_n$ remains finite. As $|\eta_0|$ decreases and the degree of flatness and homogeneity of the surface increases, the attenuation of the primary waves scattered by each cell at small grazing angles decreases. In this case the role of the "focusing" factor $\cos\theta_0/\cos\theta_n$ may become dominant, which will lead to an increase in the amplitudes of the grazing spectra. If the acoustic interaction of neighboring cells is not taken into account, and consequently the attenuation of waves grazing along the surface is neglected, then, generally speaking, we obtain $\varphi(\pm\pi/2) \neq 0$, and in the approximation considered the amplitudes of the grazing spectra will become infinite, as in (4). From this point of view, the role of multiple reflections reduces to decreasing the amplitudes of the grazing spectra to finite values by "pumping" the energy of the grazing waves into the Bragg spectra of other directions. From another point of view, the increase of the amplitude as $\theta_n \to \pi/2$ is explained by the property of the grating to focus energy more strongly in the grazing spectra, because the effective Fresnel zone for them is larger than for the other spectra.

If the surface is sufficiently flat (the assumptions of the solution (1) are satisfied), then one may set $\alpha_m(u) \simeq \alpha_m^{(1)}(u)$. The approximate solution thus obtained, which represents a parallel formulation of the method [5] for a surface that is not absolutely compliant, encompasses all qualitative features of the exact solution.

In conclusion, we note that $\alpha_m^{(2)}(u)$ can be represented in the form of an expansion in a series of elementary functions.

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CITED LITERATURE

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* In (3) this phenomenon was called "surface resonance."

Note: Figure translations are in progress. See original paper for figures.