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Abstract

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ON THE APPLICABILITY OF THE FOURIER METHOD TO THE SOLUTION OF THE FIRST BOUNDARY-VALUE PROBLEM FOR A QUASILINEAR PARABOLIC EQUATION

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In this note we consider a parabolic equation with a nonlinearity of the form

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left[p(x)\frac{\partial u}{\partial x}\right] - q(x)u + \mu f(x, t, u(x, t)). \tag{1}$$

We assume that on the interval $[0, l]$ the function $p(x)$ is twice continuously differentiable, and $q(x)$ is continuously differentiable, with $p(x) \geq p_0 > 0$. It is assumed that $f(x, t, u)$ is continuous together with its derivatives $\partial f/\partial x$, $\partial f/\partial u$, $\partial^2 f/\partial x^2$, $\partial^2 f/\partial x \partial u$, and $\partial^2 f/\partial u^2$ with respect to all its arguments in the domain $(0 \leq x \leq l,\ 0 \leq t \leq T,\ |u| < \infty)$; moreover, for $0 \leq x \leq l$, $0 \leq t \leq T$, and $|u| \leq K$, the inequalities

$$\left|\partial^k f/\partial x^m \partial u^n\right| \leq F(T)\varphi(K), \qquad k = 0, 1, 2, \quad m + n = k, \tag{2}$$

hold, where $F(T)$ and $\varphi(K)$ are nonnegative, nondecreasing functions of their arguments, defined for all positive T and K and finite for finite T and K. In addition, let $f(x, t, u)$ satisfy

$$f(0, t, u) = f(l, t, u) = 0. \tag{3}$$

The first boundary-value problem for a non-self-adjoint parabolic equation with coefficients depending on u has been considered by a number of authors ([5–8]). The purpose of our note is to apply the Fourier method and Schauder' s method to the solution of the problem posed. As a result of applying the Fourier method, the problem is reduced to an infinite system of nonlinear integral equations, whose solution is obtained by Schauder's method (the fixed-point method). Two definitions of generalized solutions are introduced; to prove their existence and uniqueness, fewer restrictions on the initial data are required than for classical solutions.

First consider the classical formulation of the first boundary-value problem for equation (1), consisting in the determination of a solution $u(x,t)$, continuous together with the derivatives $\partial u/\partial x$, $\partial u/\partial t$, $\partial^2 u/\partial x^2$ in the closed domain G $(0 \le x \le l,\ 0 \le t \le T)$, satisfying the initial condition

$$u(x,0) = g(x) \tag{4}$$

and the boundary conditions

$$u(0,t) = u(l,t) = 0. \tag{5}$$

By the Fourier method, the solution of the first boundary-value problem (1), (4), (5) is sought in the form

$$u(x,t) = \sum_{n=1}^{\infty} A_n(t) X_n(x), \tag{6}$$

which reduces the problem (1), (4), (5) to the study of the infinite system of nonlinear integral equations

$$A_n(t) = c_n e^{-\lambda_n t} + \mu \int_0^t \int_0^l e^{-\lambda_n (t-\tau)} f\left(x, \tau, \sum_{k=1}^{\infty} A_k(\tau) X_k(x)\right) X_n(x)\, d\tau\, dx, \tag{7}$$

$n = 1, 2, ...,$ where $X_n(x)$ is a complete orthonormal system of eigenfunctions of the Sturm–Liouville problem

$$L(X) + \lambda X = 0; \tag{8}$$

$$X(0) = X(l) = 0; \tag{9}$$

$$L(X) \equiv \frac{d}{dx}\left[p(x)\frac{dX(x)}{dx}\right] - q(x)X(x); \tag{10}$$

$$c_n = \int_0^l g(x) X_n(x)\, dx. \tag{11}$$

In what follows we use the well-known [1] properties of the eigenfunctions $X_n(x)$ and eigenvalues λ_n of the Sturm–Liouville problem (8), (9).

Lemma 1. If $f(x,t,u)$ satisfies (2), (3), and $u(x,t)$ is measurable in t and continuous, together with $\partial u/\partial x$, in x, and in G satisfies the inequalities

$$|u(x,t)| \le K; \tag{12}$$

$$|\partial u(x,t)/\partial x| \le K; \tag{13}$$

$$|\partial u(x+\Delta x,t)/\partial x - \partial u(x,t)/\partial x| \le K|\Delta x|, \tag{14}$$

then the Fourier coefficients of $f(x,t,u(x,t))$ with respect to the system $X_n(x)$ satisfy the inequality

$$|b_n(t)| \le D/\lambda_n, \tag{15}$$

where the constant D depends on $p(x)$, $p'(x)$, $q(x)$, K, T.

The investigation of system (7) is carried out according to the scheme of the work [(2)] of one of the authors as follows. Consider the nonmetrizable complete linear topological space $C^{(1)}$ of functions $u(x,t)$ satisfying (5), continuous in x and t in the domain G, having partial derivatives $\partial u/\partial x$, continuous in x and measurable in t, with topology given by the operation of passage to the limit

$$\lim_{k\to\infty} \sup_{(x,t)\in G} |u(x,t) - u_k(x,t)| = 0, \tag{16}$$

$$\lim_{k\to\infty} \sup_{0\le x\le l} |\partial u(x,t)/\partial x - \partial u_k(x,t)/\partial x| = 0 \tag{17}$$

(for any fixed t from $[0,T]$). In $C^{(1)}$ consider the closed convex set $C_k^{(1)}$ of elements of $C^{(1)}$ satisfying (12), (13), and (14), where K is a constant. On $C_k^{(1)}$ define a nonlinear operator A ($v = Au$) in such a way that, if $u \in C_k^{(1)}$ and is given with the aid of (6), then

$$v(x,t) = \sum_{n=1}^{\infty} B_n(t) X_n(x),$$

where $B_n(t)$ is determined with the aid of $u(x,t)$ from

$$B_n(t) = c_{ne}^{-\lambda_n t} + \mu \int_0^t \int_0^l e^{-\lambda_n(t-\tau)} f(x,\tau,u(x,\tau)) X_n(x)\, d\tau\, dx, \qquad n = 1, 2, ...$$

The range of values of the operator A then lies in $C^{(1)}$.

Theorem 1. For sufficiently small μ, the operator A maps the convex set $C_k^{(1)}$ in $C^{(1)}$ into its compact part.

Theorem 2. If $f(x,t,u)$ satisfies (2), (3), then, for μ sufficiently small, there exists at least one invariant point of the mapping $v = Au$.

Theorem 3. The infinite system (7) has a unique solution $A_n(t)$, $n = 1, 2, ...$, in the class of continuous functions satisfying the conditions $|A_n(t)| \leq Aa_n$, where $\sum_{n=1}^{\infty} a_n < \infty$, if $f(x,t,u)$ is continuous in x, t, and satisfies a Lipschitz condition in u.

Theorem 4. If $p(x), q(x), f(x,t,u)$ satisfy the requirements formulated at the beginning of the note; $g(x)$ is three times continuously differentiable and satisfies conditions (9) and $L(g) = 0$ for $x = 0$ and $x = l$, then, for sufficiently small μ, there exists in G a unique classical solution of the first boundary-value problem (1), (4), (5), representable in the form (6).

Remark. Since we impose no restrictions on the character of the behavior of $\partial f/\partial u$, it is not surprising (see (3)) that the existence and uniqueness of the classical solution of the first boundary-value problem (1), (4), (5) have been proved for small μ.

Theorem 5 (correctness). Classical solutions $u_1(x,t), u_2(x,t)$ of the first boundary-value problem for equation (1), satisfying (5), (12), and the initial conditions

$$u_1(x,0) = g_1(x); \tag{18}$$

$$u_2(x,0) = g_2(x) \tag{19}$$

for sufficiently small μ will differ in modulus by an arbitrarily small amount in G, if $|g_1(x) - g_2(x)|,\ |g_1'(x) - g_2'(x)|$ are sufficiently small for all x on $[0,l]$.

Theorem 6. If $u_1(x,t), u_2(x,t)$ are classical solutions in G of the first boundary-value problem (1), (5), representable in the form (6), satisfying (12) and the initial conditions (19), (20), then for $0 \leq t \leq T$

$$\int_0^l [u_1(x,t) - u_2(x,t)]^2\, dx \leq A \int_0^l [g_1(x) - g_2(x)]^2\, dx, \tag{20}$$

where the constant A depends only on K and T.

Definition 1. A continuous function $u(x,t)$ will be called a **generalized solution of the first boundary-value problem** (1), (4), (5), if $u(x,t)$ is the limit, uniformly convergent in G as $m \to \infty$, of a sequence of classical solutions

$u_m(x,t)$, representable in the form (6), of equation (1) with boundary conditions (5) and initial data

$$u_m(x,0) = g_m(x) \tag{21}$$

such that $g_m(x)$ converges uniformly on $[0,l]$ to $g(x)$.

Theorem 7. If $g(x)$ satisfies (9) and has $g'(x), g''(x)$, with $g''(x)$ square-integrable on $[0,l]$, then, for sufficiently small μ, there exists at least one generalized solution $u(x,t)$, in the sense of Definition 1, of the first boundary-value problem (1), (4), (5).

To prove Theorem 7, one constructs a sequence of classical solutions $u_m(x,t)$ of problem (1), (5), satisfying the initial conditions (21), where

$$g_m(x) = \sum_{k=1}^{m} c_k X_k(x)$$

(c_k are the Fourier coefficients of $g(x)$ (11)). The classical solutions are sought in the form

$$u_m(x,t) = \sum_{k=1}^{\infty} A_k^{(m)}(t) X_k(x),$$

which leads to the question of the existence of a solution of a finite system of nonlinear integral equations. A linear nonmetrizable complete topological space R of functions $u(x,t)$, defined on G, continuous in x, measurable in t, satisfying (5), is introduced, with topology specified by the limiting transition

$$\lim_{k\to\infty} \sup_{0\le x\le l} |u(x,t) - u_k(x,t)| = 0.$$

In R one considers a compact convex set D of elements u from R, satisfying (12) and $|u(x+\Delta x,t) - u(x,t)| \le L|\Delta x|$. On D

one considers the operator A_m ($v = A_m u$), defined as follows: if $u \in D$, then put

$$v(x,t) = \sum_{k=1}^{m} B_k^{(m)}(t) X_k(x),$$

where $B_k^{(m)}(t)$ is determined with the help of $u(x,t)$ from

$$B_k^{(m)}(t) = c_k e^{-\lambda_k t} + \mu \int_0^t \int_0^l e^{-\lambda_k (t-\tau)} f(x, \tau, u(x, \tau)) X_k(x)\, d\tau\, dx, \quad k = 1, 2, \dots, m.$$

The proof of the existence of a solution of the equation $u = A_m u$ in the complete linear topological, but nonmetrizable, space R reduces to the application of Schauder' s principle in a complete linear metric space by means of a device of A. N. Tikhonov ([4]).

Theorem 8 (correctness of the generalized solution). *If there exists a generalized solution, in the sense of Definition 1, of the first boundary-value problem (1), (4), (5), then it is unique in the class of functions continuous on G, provided the initial function is continuous on $[0, l]$ and satisfies (9).*

Theorem 9. *If $u_1(x, t)$, $u_2(x, t)$ are generalized, in the sense of Definition 1, solutions of the first boundary-value problem (1), (5) with initial data (18), (19), then, for μ sufficiently small, $|u_1(x, t) - u_2(x, t)|$ will be arbitrarily small in G, provided $|g_1(x) - g_2(x)|$, $|g_1'(x) - g_2'(x)|$ are sufficiently small for all x in $[0, l]$.*

Definition 2. A bounded function $u(x, t)$, for which

$$\int_0^l u^2(x, t)\, dx$$

is continuous with respect to t on $[0, T]$, will be called a **generalized solution in G of the first boundary-value problem** (1), (4), (5), if in G there exists a uniformly bounded sequence of classical solutions $u_m(x, t)$, representable in the form (6), of the first boundary-value problem (1), (5) with initial data (21) such that

$$\lim_{m \to \infty} \int_0^l [g(x) - g_m(x)]^2\, dx = 0$$

and, uniformly in t ($0 \le t \le T$),

$$\lim_{m \to \infty} \sup_{0 \le t \le T} \int_0^l [u(x, t) - u_m(x, t)]^2\, dx = 0.$$

Theorem 10. *If there exists a generalized, in the sense of Definition 2, solution $u(x, t)$ in G of the first boundary-value problem (1), (4), (5), then it is unique in the class of bounded functions $v(x, t)$ for which*

$$\int_0^l v^2(x, t)\, dx$$

is continuous with respect to t. The initial function $g(x)$ is here assumed continuous on $[0, l]$ and satisfying (9).

Theorem 11. *If $g(x)$ satisfies (9) and is continuously differentiable, then, for μ sufficiently small, there exists at least one generalized, in the sense of Definition 2, solution $u(x, t)$ of the first boundary-value problem (1), (4), (5), and moreover $|u(x, t)| \le K$. For this generalized solution the inequality (20) also holds.*

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Note: Figure translations are in progress. See original paper for figures.