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Abstract

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MATHEMATICS

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CONDITIONS FOR THE CLOSEDNESS OF A RESTRICTION OF A SELF-ADJOINT OPERATOR

(Presented by Academician A. N. Kolmogorov on 21 I 1960)

In the present paper a description is given of closed operators with domain dense in a Hilbert space and contained in a given self-adjoint operator. For simplicity of exposition we shall restrict ourselves to the case of a simple spectrum. However, the method of description presented is applicable to closed restrictions of any self-adjoint, or, more generally, normal operator with finite-multiplicity spectrum.

As is known, a self-adjoint operator with simple spectrum is unitarily equivalent to the operator Λ_σ of multiplication by the independent variable in a certain space L_σ^2 , where σ is a nonnegative measure given on the real axis;

$$L_\sigma^2 = \left\{ x : \int_{-\infty}^{\infty} |x(\lambda)|^2 d\sigma(\lambda) < \infty \right\};$$

$$\mathfrak{D}_{\Lambda_\sigma} = \left\{ x : x \in L_\sigma^2, \int_{-\infty}^{\infty} |\lambda x(\lambda)|^2 d\sigma(\lambda) < \infty \right\}, \quad (\Lambda_\sigma x)(\lambda) = \lambda x(\lambda).$$

Since closedness and density are invariant under unitary transformations, the problem reduces to the description of operators A , acting from L_σ^2 into L_σ^2 , such that $\mathfrak{D}_A = L_\sigma^2$, $A^{**} = A$, and $A \subset \Lambda_\sigma$.

For stating the results obtained, denote by M_σ the collection of all σ -measurable and σ -almost everywhere finite complex-valued functions defined on the real axis. We have $L_\sigma^2 \subset M_\sigma$. Introduce the operation $^\perp$, assigning to each subset $U \subset M_\sigma$ the linear manifold $U^\perp \subset L_\sigma^2$ by the formula

$$U^\perp = \left\{ v : v \in L_\sigma^2, \int u(\lambda) \overline{v(\lambda)} d\sigma(\lambda) \text{ exists and is equal to 0 for all } u \in U \right\}.$$

Theorem 1. For every operator $A \in \Lambda_\sigma$, $\overline{\mathfrak{D}}_A = L_\sigma^2$, $A^{**} = A$, there exists a linear manifold $U \subset M_\sigma$ such that

$$\mathfrak{D}_A = \mathfrak{D}_{\Lambda_\sigma} \cap U^\perp; \quad (1)$$

moreover, U contains no functions from L_σ^2 (except the zero function), i.e.

$$\int_{-\infty}^{\infty} |u(\lambda)|^2 d\sigma(\lambda) = \infty, \quad u \neq 0, \quad u \in U, \quad (2)$$

but for each function $u \in U$

$$\int_{-\infty}^{\infty} \frac{|u(\lambda)|^2}{1 + \lambda^2} d\sigma(\lambda) < \infty. \quad (3)$$

It is unknown to the author whether, for a given A , formula (1) determines the linear manifold $U \subset M_\sigma$ uniquely. Therefore, it may be that relations (2) and (3) are not necessary. Moreover, it is unknown whether relations (2) and (3) are sufficient in order that the relations $\overline{\mathfrak{D}}_A = L_\sigma^2$, $A^{**} = A$ should hold, if \mathfrak{D}_A is given by formula (1) and $A \in \Lambda_\sigma$. However, under certain sufficiently general assumptions, relations (2) and (3) are necessary and sufficient.

In order to formulate these assumptions, introduce the operation¹ which assigns to a set $V \subset L_\sigma^2$ the linear manifold $V^\perp \subset M_\sigma$ by the formula

$$V^\perp = \left\{ u : u \in M_\sigma, \int u(\lambda) \overline{v(\lambda)} d\sigma(\lambda) \text{ exists and is equal to 0 for all } v \in V \right\}.$$

We shall call a manifold $U \subset M_\sigma$ \perp -closed if

$$(\mathfrak{D}_{\Lambda_\sigma} \cap U^\perp)^\perp = U, \quad (4)$$

and strongly \perp -closed if

$$(\mathfrak{D}_{\Lambda_\sigma} \cap U_1^\perp)^\perp = U_1 \quad (5)$$

for every linear manifold $U_1 \subset U$.

Lemma 1. Every finite-dimensional linear manifold is strongly \perp -closed and, a fortiori, \perp -closed.

Lemma 2. Let a linear manifold $U \subset M_\sigma$ have the following properties:

α) if, for some function y , there exists an increasing sequence of σ -measurable sets $\mathcal{E}_1, \mathcal{E}_2, \dots$ with union $\bigcup \mathcal{E}_n$ of full σ -measure, on each of which the function y is equal to some function from U , then $y \in U$;

β) there exists an increasing sequence of σ -measurable sets $\mathcal{E}'_1, \mathcal{E}'_2, \dots$, whose union has full σ -measure and on each of which every function $u \in U$ is square-integrable with respect to the measure σ .

Then the manifold U is \perp -closed.*

Example of an infinite-dimensional manifold U , strongly \perp -closed: the set of (finite) linear combinations of the functions $u_\alpha(\lambda) = \lambda^\alpha$, $-1/2 \leq \alpha < 1/2$; σ is Lebesgue measure.

Theorem 2. Let a linear manifold $U \subset M_\sigma$ be \perp -closed. Let $A \subset \Lambda_\sigma$, $\mathfrak{D}_A = \mathfrak{D}_{\Lambda_\sigma} \cap U^\perp$. If every function $u \in U$, $u \neq 0$, satisfies relation (2), then $\overline{\mathfrak{D}}_A = L^2_\sigma$. If, in addition, every function $u \in U$ satisfies condition (3), then $A^{**} = A$.

Theorem 3. Let a linear manifold $U \subset M_\sigma$ be strongly \perp -closed. Let $A \subset \Lambda_\sigma$ and $\mathfrak{D}_A = \mathfrak{D}_{\Lambda_\sigma} \cap U^\perp$. If $\overline{\mathfrak{D}}_A = L^2_\sigma$, $A^{**} = A$, then every function $u \in U$ satisfies relations (2) and (3).

* Under the conditions of Lemmas 1 and 2, $(\mathfrak{D} \cap U^\perp)^\perp = U$, where \mathfrak{D} is the domain of definition of any function of the operator Λ_σ .

Proof of Theorem 1. Let $\mathfrak{D}_A = L^2_\sigma$ and $A \subset \Lambda_\sigma$. Then A is a symmetric operator: $A \subset A^*$, $\mathfrak{D}_{A^*} = L^2_\sigma$, and therefore A^{**} exists. It is not hard to verify that

$$\mathfrak{D}_{A^{**}} = \mathfrak{D}_{\Lambda_\sigma} \cap \left[(A^* - \tilde{\Lambda}) \mathfrak{D}_{A^*} \right]^\perp, \quad (1')$$

where $(\tilde{\Lambda}x)(\lambda) = \lambda x(\lambda)$, by definition, for every function $x \in M_\sigma$. Therefore, if $A^{**} = A$, then, putting

$$U = (A^* - \tilde{\Lambda}) \mathfrak{D}_{A^*}, \quad (6)$$

we obtain formula (1): $\mathfrak{D}_A = \mathfrak{D}_{\Lambda_\sigma} \cap U^\perp$.

If at least one $u \neq 0$ from U belongs to L^2_σ , then U^\perp is not dense in L^2_σ . Therefore relation (2) holds. In accordance with (6), for every $u \in U$ there exists such a $y \in \mathfrak{D}_{A^*}$ that $\tilde{\Lambda}y + u = A^*y$. Since $A^*y \in L^2_\sigma$, it follows that

$$\int |\lambda y(\lambda) + u(\lambda)|^2 d\sigma(\lambda) < \infty. \quad (7)$$

From $y \in L_\sigma^2$ it follows that

$$\int_{|\lambda| < \varepsilon} |\lambda y(\lambda)| \sigma(d\lambda) < \infty, \quad \varepsilon > 0. \quad (8)$$

From (7) and (8) we derive

$$\int_{|\lambda| < \varepsilon} |u(\lambda)|^2 d\sigma(\lambda) < \infty. \quad (9)$$

Moreover, from (7) we obtain

$$\int_{|\lambda| \geq \varepsilon} \left| y(\lambda) + \frac{u(\lambda)}{\lambda} \right|^2 d\sigma(\lambda) < \infty,$$

whence, since $y \in L_\sigma^2$,

$$\int_{|\lambda| \geq \varepsilon} \left| \frac{u(\lambda)}{\lambda} \right|^2 d\sigma(\lambda) < \infty. \quad (10)$$

Relations (9) and (10), taken together, are equivalent to relation (3).

Proof of Theorem 2. Suppose that U is \perp -closed. If some element $x \in L_\sigma^2$ is orthogonal to all elements of $\mathfrak{D}_{\Lambda_\sigma} \cap U^\perp$, then $x \in (\mathfrak{D}_{\Lambda_\sigma} \cap U^\perp)^\perp = U$ (see (4)), i.e. $x \in L_\sigma^2 \cap U$. If condition (2) is satisfied, then $x = 0$ and $\mathfrak{D}_A = \mathfrak{D}_{\Lambda_\sigma} \cap U^\perp$ is dense in L_σ^2 . Let us verify that, under our assumptions:

γ) the domain \mathfrak{D}_{A^*} of the operator A^* consists exactly of those $y \in L_\sigma^2$ for which there exists such a $u \in U$ that $\tilde{\Lambda}y + u \in L_\sigma^2$, and in this case $\tilde{\Lambda}y + u = A^*y$.

Let $x \in \mathfrak{D}_A$, $y \in \mathfrak{D}_{A^*}$. Put $u = (A^* - \tilde{\Lambda})y$. The relation $(Ax, y) = (x, A^*y)$ can be rewritten in the form

$$\int x(\lambda) \overline{u(\lambda)} d\sigma(\lambda) = 0.$$

Hence $u \in \mathfrak{D}_A^\perp = (\mathfrak{D}_{\Lambda_\sigma} \cap U^\perp)^\perp = U$ (see (4)). Conversely, suppose that for some $y \in L_\sigma^2$, $u \in U$, we have $\tilde{\Lambda}y + u \in L_\sigma^2$. Since

$$\int x(\lambda)\overline{u(\lambda)} d\sigma(\lambda) = 0$$

for $x \in \mathfrak{D}_A$, it follows that $(x, \tilde{\Lambda}y + u) = (\tilde{\Lambda}x, y) = (Ax, y)$ for all $x \in \mathfrak{D}_A$, and assertion γ) is proved.

From assertion γ) it follows that

$$(A^* - \tilde{\Lambda})\mathfrak{D}_{A^*} \subset U. \quad (11)$$

Let us verify that from relation (3) the reverse inclusion follows:

$$U \subset (A^* - \tilde{\Lambda})\mathfrak{D}_{A^*}. \quad (12)$$

Let $u \in U$. Put $y(\lambda) = 0$ for $|\lambda| < \varepsilon$ and $y(\lambda) = -u(\lambda)/\lambda$ for $|\lambda| \geq \varepsilon$, $\varepsilon > 0$. Relation (3) is equivalent to (9) and (10). From (10) it follows that $y \in L_\sigma^2$. We have $\tilde{\Lambda}y + u = v$, where $v(\lambda) = 0$ for $|\lambda| \geq \varepsilon$ and $v(\lambda) = u(\lambda)$ for $|\lambda| < \varepsilon$. By virtue of (9), $v \in L_\sigma^2$. In accordance with assertion γ), $y \in \mathfrak{D}_{A^*}$, $\tilde{\Lambda}y + \mu = A^*y$, $u = (A^* - \tilde{\Lambda})y \in (A^* - \tilde{\Lambda})\mathfrak{D}_{A^*}$. Inclusion (12) is proved. From (11), (12), and (1) it follows that

$$\mathfrak{D}_{A^{**}} = \mathfrak{D}_{\Lambda_\sigma} \cap [(A^* - \tilde{\Lambda})\mathfrak{D}_{A^*}]^\perp = \mathfrak{D}_{\Lambda_\sigma} \cap U^\perp = \mathfrak{D}_A.$$

Consequently, $A^{**} = A$.

Proof of Theorem 3 is similar to the proof of Theorem 1.

Proof of Lemma 2. Obviously, $U \subset (\mathfrak{D}_{\Lambda_\sigma} \cap U^\perp)^\top$, and therefore it is sufficient to prove that the inclusion sign can be directed in the opposite direction. Consider an arbitrary element

$$w \in (\mathfrak{D}_{\Lambda_\sigma} \cap U^\perp)^\top$$

and put

$$\mathcal{E}_k = \left\{ \lambda : |\lambda| < k, \frac{|w(\lambda)|}{s(\lambda)} < k \right\} \cap \mathcal{E}'_k,$$

where $\mathcal{E}'_1, \mathcal{E}'_2, \dots$ is a sequence of sets possessing the property described in condition β), and s is some almost everywhere positive function from L_σ^2 : $s(\lambda) > 0$, $\int [s(\lambda)]^2 d\sigma(\lambda) < \infty$. Denote by χ_k the characteristic function of the set \mathcal{E}_k , and also the operator of multiplication by this function in M_σ . Put $L_\sigma^2(\mathcal{E}_k) = \chi_k L_\sigma^2$: all functions from $L_\sigma^2(\mathcal{E}_k)$ are equal to zero outside \mathcal{E}_k and $L_\sigma^2(\mathcal{E}_k) \subset L_\sigma^2$. Moreover,

$$\chi_k U \subset L_\sigma^2(\mathcal{E}_k) \subset \mathfrak{D}_{\Lambda_\sigma}, \quad (13)$$

for every function $u \in U$ is square integrable on each of the sets \mathcal{E}_k (see condition β)) and $|\lambda| < k$ on \mathcal{E}_k . Therefore, as is not difficult to verify,

$$L^2_\sigma(\mathcal{E}_k) \cap U^\perp = L^2_\sigma(\mathcal{E}_k) \ominus \chi_k U. \quad (14)$$

Obviously, $w \in (L^2_\sigma(\mathcal{E}_k) \cap U^\perp)^\top$, since, by (13), $L^2_\sigma(\mathcal{E}_k) \cap U^\perp$ is contained in $\mathfrak{D}_{\Lambda_\sigma} \cap U^\perp$. Consequently, also

$$\chi_k w \in (L^2_\sigma(\mathcal{E}_k) \cap U^\perp)^\top.$$

But, in accordance with the definition of the set \mathcal{E}_k , $\chi_k w \in L^2_\sigma(\mathcal{E}_k)$:

$$\int_{\mathcal{E}_k} |w(\lambda)|^2 d\sigma(\lambda) \leq k^2 \int_{\mathcal{E}_k} [s(\lambda)]^2 d\sigma(\lambda).$$

Therefore, by virtue of (14), $\chi_k w \in \chi_k U$. This means that on each of the sets \mathcal{E}_k the function w is equal to some function from U . Taking into account condition α), we obtain $w \in U$, since the complement of the set $\bigcup \mathcal{E}_k$ has σ -measure zero.

Proof of Lemma 1. Every finite-dimensional linear manifold $U \subset M_\sigma$ satisfies the conditions of Lemma 2.

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Note: Figure translations are in progress. See original paper for figures.

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