Mathematics

1960

SovietRxiv

Abstract

Full Text

Mathematics

I. I. Bavrin

Estimates of the Taylor Coefficients of Functions of Several Complex Variables

(Presented by Academician M. A. Lavrent' ev on 22 XII 1959)

In the classical theory of functions of one complex variable, an important role is played by two rather general classes of functions closely connected with one another: the class of Schur functions and the class of Carathéodory functions. The problem of obtaining exact estimates for the Taylor coefficients in the last two classes of functions in the case of more than one complex variable was considered by me in papers ([1],[2]), where this problem was solved only in the case of two complex variables. In the present note, by singling out a special class of domains in the space of n complex variables, exact estimates will be given for the Taylor coefficients of the above-mentioned classes of functions in the case of n complex variables. The guiding idea in singling out the special class of domains in the space of many complex variables was the parametric representation, given by A. A. Temlyakov ([3]), of the boundary of a bounded complete bicircular domain with center at the point $(0,0)$ of the space of two complex variables, the boundary of which is twice continuously differentiable and analytically convex from the outside.

Definition. Let the domain $D \ni (0,\dots,0)$ be a circular domain in the space of n complex variables $z_1,\dots,z_n$, which is a subdomain of some domain of regularity, bounded by the nonanalytic $(2n-1)$-dimensional surface

$$
\begin{aligned}
|z_1| &= r_1(\tau_1)\dots r_1(\tau_{n-1}),\\
|z_2| &= r_2(\tau_1)r_1(\tau_2)\dots r_1(\tau_{n-1}),\\
|z_3| &= r_2(\tau_2)r_1(\tau_3)\dots r_1(\tau_{n-1}),\\
&\dots\dots\dots\dots\dots\\
|z_i| &= r_2(\tau_{i-1})r_1(\tau_i)\dots r_1(\tau_{n-1}),\\
&\dots\dots\dots\dots\dots\\
|z_{n-1}| &= r_2(\tau_{n-2})r_1(\tau_{n-1}),\\
|z_n| &= r_2(\tau_{n-1}),\\
0 \le \tau_i &\le 1, \qquad i = 1,2,\dots,n-1,
\end{aligned}
$$

where $r_1(0)=0$, $r_1'(\tau)>0$ in $(0,1]$, $r_1(1)<\infty$, and

$$r_2(\tau) = \exp\left[-\int \frac{\tau}{1-\tau}\, d\ln r_1(\tau)\right]$$

($r_2(1) = 0$ ([3])).

A special case of the domain D will be a domain containing its center $(0, \dots, 0)$ and bounded by the $(2n-1)$-dimensional surface

$$\begin{gathered}
|z_1| = R(\tau_1 \dots \tau_{n-1})^\alpha, \\
|z_2| = R[(1-\tau_1)\tau_2 \dots \tau_{n-1}]^\alpha, \\
|z_3| = R[(1-\tau_2)\tau_3 \dots \tau_{n-1}]^\alpha, \\
\dots\dots\dots\dots \\
|z_i| = R[(1-\tau_{i-1})\tau_i \cdots \tau_{n-1}]^\alpha, \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
|z_{n-1}| = R[(1-\tau_{n-2})\tau_{n-1}]^\alpha, \\
|z_n| = R(1-\tau_{n-1})^\alpha, \\
0 \le \tau_i \le 1, \qquad i = 1, 2, \dots, n-1, \qquad 0 < \alpha \le 1,
\end{gathered}$$

R is a positive constant, i.e., a domain of the form

$$|z_1|^{1/\alpha} + \dots + |z_n|^{1/\alpha} < R^{1/\alpha}.$$

We shall denote this domain by C. If $\alpha = 1$, then we have the hypercone

$$|z_1| + \dots + |z_n| < R;$$

if $\alpha = 1/2$, the hypersphere

$$|z_1|^2 + \dots + |z_n|^2 < R^2.$$

Lemma 1. If the function

$$F(z_1, \dots, z_n) = \sum_{m_1, \dots, m_n = 0}^{\infty} a_{m_1 \dots m_n} z_1^{m_1} \cdots z_n^{m_n},$$

where $a_{0\dots0}$ is given, is regular in the polycylinder

$$S\{|z_m| < R_m,\ m = 1, 2, \dots, n\}$$

and is such that $|F(z_1, \dots, z_n)| < 1$ in S, then for $m_1 + \dots + m_n > 0$

$$|a_{m_1 \dots m_n}| \le \frac{1 - |a_{0\dots0}|^2}{R_1^{m_1} \dots R_n^{m_n}}. \tag{1}$$

The proof is based on Cauchy' s formula in the case of a polycylinder and on Cauchy' s theorem for functions of one complex variable.

Theorem 1. If the function

$$F(z_1, \dots, z_n) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1\dots m_n} z_1^{m_1} \cdots z_n^{m_n},$$

where $a_{0\dots0}$ is given, is regular in the domain D and satisfies in D the condition

$$|F(z_1, \dots, z_n)| < 1,$$

then for $m_1 + \dots + m_n > 0$

$$|a_{m_1\dots m_n}| \leq \frac{1 - |a_{0\dots0}|^2}{M}, \tag{2}$$

where

$$M = \prod_{i=1}^{n-1} r_1^{\sum_{k=1}{}^{i} m_k} \left(\frac{\sum_{k=1}^{i} m_k}{\sum_{k=1}^{i+1} m_k} \right) r_2^{m_i+1} \left(\frac{\sum_{k=1}^{i} m_k}{\sum_{k=1}^{i+1} m_k} \right),$$

with $0^0 = 1$.

In the proof, Lemma 1 is used, as well as the parametric representation of the boundary of the domain D and the maximum of the product

$$\prod_{i=1}^{n-1} r_1^{\sum_{k=1}{}^{i} m_k}(\tau_i)\, r_2^{m_i+1}(\tau_i)$$

over

$$\{0 \leq \tau_i \leq 1,\ i = 1, 2, \dots, n-1\}.$$

Remark 1. The estimates (1), (2) are sharp, since they are attained by the functions

$$F(z_1, \dots, z_n) \equiv \frac{a_{0\dots0} R_1^{m_1} \cdots R_n^{m_n} + q z_1^{m_1} \cdots z_n^{m_n}}{R_1^{m_1} \cdots R_n^{m_n} + \bar{a}_{0\dots0} q z_1^{m_1} \cdots z_n^{m_n}},$$

$$F(z_1, \dots, z_n) \equiv \frac{a_{0\dots0} M + q z_1^{m_1} \cdots z_n^{m_n}}{M + \bar{a}_{0\dots0} q z_1^{m_1} \cdots z_n^{m_n}},$$

where $|a_{0\dots0}| < 1,\ |q| = 1$.

Remark 2. We note that in the case of the domain C

$$M = \frac{m_1^{\alpha m_1} \cdots m_n^{\alpha m_n}}{(m_1 + \dots + m_n)^{\alpha(m_1+\dots+m_n)}} R^{m_1+\dots+m_n}.$$

Corollary. Let, in the domain D, the function

$$F(z_1, \dots, z_n) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1\dots m_n} z_1^{m_1} \cdots z_n^{m_n}$$

be regular. Then, if in D, $|F(z_1, \dots, z_n)| < 1$, then for $m_1 + \dots + m_n \geqslant 0$

$$|a_{m_1\dots m_n}| \leqslant \frac{1}{M}. \tag{3}$$

Remark 3. Estimate (3) is sharp, since equality in (3) occurs for the function

$$F(z_1, \dots, z_n) = \frac{z_1^{m_1} \cdots z_n^{m_n}}{M}.$$

Lemma 2. If in the polycylinder S the function

$$F(z_1, \dots, z_n) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1\dots m_n} z_1^{m_1} \cdots z_n^{m_n} \qquad (a_{0\dots 0} = 1)$$

is regular and $\operatorname{Re} F(z_1, \dots, z_n) > 0$, then for $m_1 + \dots + m_n > 0$

$$|a_{m_1\dots m_n}| \leqslant \frac{2}{R_1^{m_1} \cdots R_n^{m_n}}. \tag{4}$$

The proof follows from Cauchy’ s formula in the case of a polycylinder and Cauchy’ s theorem for functions of one complex variable.

In the same way as Theorem 1, but using Lemma 2, one proves

Theorem 2. If in the domain D the function

$$F(z_1, \dots, z_n) = \sum_{m_1,\dots,m_n=0}^{\infty} a_{m_1\dots m_n} z_1^{m_1} \cdots z_n^{m_n} \qquad (a_{0\dots 0} = 1)$$

is regular and $\operatorname{Re} F(z_1, \dots, z_n) > 0$, then for $m_1 + \dots + m_n > 0$

$$|a_{m_1\dots m_n}| \leqslant \frac{2}{M}. \tag{5}$$

Remark 4. Estimates (4) and (5) are sharp, since they are attained by the functions

$$F(z_1, \dots, z_n) = \frac{R_1^{m_1} \cdots R_n^{m_n} + z_1^{m_1} \cdots z_n^{m_n}}{R_1^{m_1} \cdots R_n^{m_n} - z_1^{m_1} \cdots z_n^{m_n}},$$

$$F(z_1, \dots, z_n) = \frac{M + z_1^{m_1} \cdots z_n^{m_n}}{M - z_1^{m_1} \cdots z_n^{m_n}}.$$

Moscow Regional Pedagogical Institute
named after N. K. Krupskaya

Received
17 XII 1959

References

1. I. I. Bavrin, DAN, **126**, No. 5 (1959).
2. I. I. Bavrin, *Uch. zap. Mosk. obl. ped. inst.*, **77**, 53 (1959).
3. A. A. Temlyakov, DAN, **120**, No. 5 (1958).

Note: Figure translations are in progress. See original paper for figures.