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Abstract

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ON THE STRUCTURE OF MACH LINES IN RELAXING MEDIA

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1. As is known ([1]), the characteristics of the equations of relaxation hydrodynamics ([2]) do not coincide with the characteristics of equilibrium hydrodynamics for all values of the relaxation time τ. On the other hand, for sufficiently small τ both systems of equations must lead to the same physical consequences. The resolution of the question lies in the fact that the role of characteristics in the formulation of the laws of motion of a medium is different in these two cases. In equilibrium hydrodynamics they determine the laws of propagation of small disturbances and, for a plane steady flow, coincide with the Mach lines. To clarify the properties of experimentally observed Mach lines from the point of view of relaxation hydrodynamics, let us consider the flow past a thin wedge (apex angle $2\alpha \ll 1$) at zero angle of attack.

Let the x-axis coincide with the direction of motion of the undisturbed flow; v is the velocity component along the u-axis arising as a result of the disturbances introduced by the wedge. Then it follows from the equations of relaxation hydrodynamics ([2]) that

$$l\frac{\partial}{\partial x}\left\{(M_\infty^2-1)\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 v}{\partial y^2}\right\}+(M_0^2-1)\frac{\partial^2 v}{\partial x^2}-\frac{\partial^2 v}{\partial y^2}=0, \tag{1}$$

where

$$M_\infty=\frac{u}{c_\infty},\qquad M_0=\frac{u}{c_0},\qquad l=u\tau\frac{c_\infty^2}{c_0^2}; \tag{2}$$

c_∞ is the speed of sound at frequency $\omega \gg 1/\tau$; c_0 is the equilibrium speed of sound ($\omega \ll 1/\tau$); u is the velocity of the undisturbed flow. Completely analogous equations can be obtained for disturbances of other hydrodynamic quantities (density, temperature, etc.).

For $M_\infty > 1$, equation (1) is of hyperbolic type with characteristic directions

$$x=\pm\sqrt{M_\infty^2-1}\,y+\text{const}. \tag{3}$$

In what follows we shall consider only the case $M_\infty > 1$. The boundary conditions have the form

$$v(x,0) = \begin{cases} 0, & \text{for } x < 0, \\ \alpha u, & \text{for } x > 0. \end{cases} \tag{4}$$

2. Since $v(x,0)$ does not vanish as $x \to \infty$, instead of an expansion in a Fourier integral it is expedient to use a representation of this function in the form of a contour integral

$$v(x,0) = \frac{\alpha u}{2\pi i} \int \frac{e^{i\xi x}}{\xi}\, d\xi : \tag{5}$$

The solution of equation (1) under the given boundary conditions is expressed in the form

$$v(x,y) = \frac{\alpha u}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{i\xi\{x' - c(\xi) y'\}}}{\xi}\, d\xi. \tag{6}$$

where

$$c(\xi) = \sqrt{\frac{i(M_\infty^2 - 1)\xi + (M_0^2 - 1)}{i\xi + 1}}, \tag{7}$$

$$x' = \frac{x}{l}, \qquad y' = \frac{y}{l}. \tag{8}$$

The integration in (5) and (6) is carried out along the positive direction of the real axis, bypassing the point $\xi = 0$ in the lower half-plane. For $y > 0$ it is necessary to retain only that value of the square root in (7) for which $\operatorname{Re} c(\xi) > 0$, since the negative values correspond to disturbances propagating upstream.

By deforming the contour of integration, it is easy to show that for

$$y > \frac{1}{\sqrt{M_\infty^2 - 1}}\, x$$

$v(x,y)$ vanishes. Thus the characteristic

$$x - \sqrt{M_\infty^2 - 1}\, y = 0, \tag{9}$$

passing through the vertex of the wedge, separates the undisturbed flow from the disturbed one.

Let us consider the behavior of the solution along characteristic (9). The contour integral (6) can be transformed into an integral over a circle of large radius centered at the origin[3]. Expanding $c(\xi)$ in a series for large ξ and retaining the first terms of this series, after integration we obtain

$$v(x,y) = \begin{cases} 0, & \text{for } y > \dfrac{1}{\sqrt{M_\infty^2 - 1}}\,x, \\ \alpha u e^{-\lambda^2 y/l} I_0(z), & \text{for } y < \dfrac{1}{\sqrt{M_\infty^2 - 1}}\,x, \end{cases} \tag{10}$$

where

$$\lambda^2 = \frac{1}{2}\frac{M_0^2 - M_\infty^2}{\sqrt{M_\infty^2 - 1}} > 0; \tag{11}$$

$I_0(z)$ is the Bessel function of imaginary argument z; z is proportional to the square root of

$$\frac{x - \sqrt{M_\infty^2 - 1}\,y}{l},$$

and, consequently, near characteristic (9) $I_0(z) \simeq 1$.

Thus, near the vertex of the wedge ($y/l \ll 1,\ x/l \ll 1$) a discontinuity arises. The intensity of this discontinuity, located along the characteristic passing through the vertex of the wedge, decreases exponentially with distance from the vertex, and at distances large in comparison with $l \simeq ut$, the discontinuity practically disappears.

3. Thus, for $x \gg l,\ y \gg l$ it is necessary to consider the region far from characteristic (9). It is easy to verify that for the branch under consideration $c(\xi)$ $(y > 0)$, $\xi \operatorname{Im} c(\xi) < 0$ for all real ξ. Thus,

$$\exp\left\{-\frac{i\xi c(\xi) y}{l}\right\}$$

contains an exponentially decaying factor. Po-

therefore, for $y \gg l$, only small ξ ($\lesssim l/y$) contribute to integral (6). Expanding $c(\xi)$ in a series near $\xi = 0$:

$$c(\xi) = \sqrt{M_0^2 - 1} - \frac{M_0^2 - M_\infty^2}{\sqrt{M_0^2 - 1}}\,\frac{i\xi}{2} + \ldots \tag{12}$$

and retaining only the first term of this expansion, we represent $v(x,y)$ in the following form:

$$v(x,y) = \begin{cases} 0, & \text{for } y > \dfrac{1}{\sqrt{M_0^2 - 1}}\, x, \\ \alpha u, & \text{for } y < \dfrac{1}{\sqrt{M_0^2 - 1}}\, x. \end{cases} \tag{13}$$

This result coincides with that which follows from equilibrium hydrodynamics. However, from the point of view of relaxation hydrodynamics it appears only in the given approximation. In reality, solution (6) contains no discontinuity along the line $x - \sqrt{M_0^2 - 1}\, y = 0$. To see this, it is sufficient to take into account two terms of expansion (12). In this case the solution proves to be continuous and is expressed in terms of the probability integral

$$v(x,y) = \frac{\alpha u}{2} \left\{ \Phi\left(\frac{x - \sqrt{M_0^2 - 1}\, y}{\sqrt{\beta^2 l y}} \right) + 1 \right\}, \tag{14}$$

where

$$\Phi(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-z^2/2}\, dz, \tag{15}$$

$$\beta^2 = \frac{M_0^2 - M_\infty^2}{\sqrt{M_0^2 - 1}} > 0. \tag{16}$$

Equation (14) determines the structure of the Mach line, i.e., of the disturbance observed along the direction

$$y = \frac{1}{\sqrt{M_0^2 - 1}}\, x$$

at large (in comparison with l) distances. The width of the Mach line is thus proportional to $\sqrt{y}$:

$$\delta = \sqrt{\beta^2 l y} \tag{17}$$

and the relative width

$$\frac{\delta}{y} = \sqrt{\beta^2 \frac{l}{y}} \tag{18}$$

decreases with distance. Therefore, at large distances this region may approximately be regarded as a "line."

4. The flow considered above can evidently be interpreted as a weak shock wave arising near a wedge in a supersonic flow past it. In [4] it was shown that in stationary one-dimensional flows discontinuities of only sufficiently large intensity can exist. In [3] it was shown that an arising discontinuity of small intensity disappears exponentially with time and, for $t \gg \tau$, gives way to a continuous flow.

Using the example of the problem considered in the present work, one can see that this assertion does not extend to the behavior of shock waves near obstacles. Thus, near the vertex of the wedge, in a region of order l ($\simeq u\tau$), there exists a stationary discontinuity of arbitrarily small intensity. However, this discontinuity decays exponentially with distance from the wall.

5. Let us next consider the general problem of disturbances in a plane steady flow. Let the undisturbed flow be directed along the positive direction of the x-axis, and let F be one of the hydrodynamic quantities (velocity, density, etc.). Suppose that at $x = 0$ a stationary disturbance of the quantity F is specified, of the form $f(x)$. It is necessary to determine the disturbance ΔF of the quantity F everywhere in the x, y plane. As is known, equilibrium hydrodynamics leads in this case to the wave equation for ΔF, whose solution for $y > 0$ has the form

$$\Delta F = f\left(x - \sqrt{M^2 - 1}\, y\right), \tag{19}$$

where M is the Mach number. The solution with the opposite sign of the root is rejected on physical grounds.

If relaxation processes are taken into account, then the equation for ΔF has the form of equation (1). Put $u > c_\infty$ and write the solution of this equation in the form of a Fourier integral

$$\Delta F(x', y') = \frac{1}{\sqrt{2\pi}} \int g(\xi) e^{i\xi\{x' - c(\xi) y'\}}\, d\xi, \tag{20}$$

where

$$g(\xi) = \frac{1}{\sqrt{2\pi}} \int f(x') e^{-i\xi x'}\, dx'; \tag{21}$$

$c(\xi)$, x', y' have the same meaning as in (7) and (8). Owing to the presence in the integrand of exponentially decaying factors, for $x' = x/l \gg 1$, $y' = y/l \gg 1$ only the low-frequency components contribute to the integral. Using expansion (12) (two terms of the series) and applying the convolution theorem, after some transformations one can obtain the following expression for F:

$$\Delta F(x', y') = \frac{1}{\sqrt{2\pi}} \int f\left(x' - \sqrt{M_0^2 - 1}\, y' - \eta\right) \frac{e^{-\eta^2/2y'\beta^2}}{\sqrt{\beta^2 y'}}\, d\eta \qquad (22)$$

(β^2, see equation (16)). This representation of the solution is valid for $x \gg l$, $y \gg l$. From (22) follow the characteristic features of the propagation of small disturbances in relaxation hydrodynamics.

In equilibrium hydrodynamics the disturbance at a point $P(x, y)$ is determined by its value at a certain definite point P_0 of the x-axis, whose coordinate x_0 is equal to $x - \sqrt{M_0^2 - 1}\, y$. By contrast, in relaxation hydrodynamics it is composed of disturbances arising at different points of the x-axis, whose contributions decrease exponentially with distance from P_0.

It also follows from (22) that to distances $y \gg l$ there is transmitted not the exact value of the disturbance that arose on the x-axis, but a certain averaged value of the initial disturbance. As a result of the integration, "fine" details of $f(x)$ gradually disappear.

6. Another feature of the propagation of small disturbances in relaxing media follows from the results (10), (11). Only at small distances ($\ll l$) do small disturbances propagate along characteristics. At large distances ($\gg l$) the propagation of disturbances is determined by the equilibrium speed of sound c_0; however, according to (22), this quantity is connected with the laws of propagation of small disturbances in a different way than is assumed by equilibrium hydrodynamics.

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Note: Figure translations are in progress. See original paper for figures.