ON THE SOLUTION OF CERTAIN BOUNDARY-VALUE PROBLEMS FOR PARABOLIC AND ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

Au+cu=f,

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Abstract

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MATHEMATICS

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ON THE SOLUTION OF CERTAIN BOUNDARY-VALUE PROBLEMS FOR PARABOLIC AND ELLIPTIC EQUATIONS WITH DISCONTINUOUS COEFFICIENTS

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1. Let G be a bounded domain in n-dimensional Euclidean space with boundary γ_0, divided by smooth $(n-1)$-dimensional manifolds γ_{ik} into subdomains G_i, $i = 1, \dots, N$. In the works of O. A. Oleinik, the existence and uniqueness of the solution of the basic boundary-value problems for the equations

$$Au + cu = f,$$

$$\frac{\partial u}{\partial t} - Au - cu = f$$

were studied under

$$Au = \sum_{i,j} \frac{\partial}{\partial x_i} a_{ij} \frac{\partial u}{\partial x_j} + b_i \frac{\partial u}{\partial x_i},$$

where $c \leq 0$; $\sum a_{ij}\xi_i\xi_j \geq \alpha \sum \xi_i^2$, $\alpha > 0$; a_{ij}, b_i, c, f are sufficiently smooth functions everywhere except at the points γ_{ik}, where these coefficients and their derivatives may have discontinuities of the first kind. The solutions were obtained as limits of classical solutions u^h for smoothed coefficients and satisfied on the lines γ_{ik} the condition $a_i \partial u/\partial \nu_i = a_k \partial u/\partial \nu_k$, where ν_i is the conormal to γ_{ik} in G_i; a_i is a positive function. In the theory of diffusion processes equations with discontinuous coefficients arise naturally, and moreover

$$Au = \sum_{i,j} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i}. \quad (1)$$

Repeating in its main features the methods of work (^{1}), it is possible also in this case to carry out the limiting passage along a sequence of solutions of equations of the form (1) with smoothed coefficients. The limiting function satisfies (1) in $\bigcup_{i=1}^{N} G_i$, and on γ_{ij} the condition of continuity of the gradient.

We shall denote by C_k^0 the class of functions having k continuous derivatives in $\bigcup_{i=1}^{N} G_i$, and by $\overline{C}_k^0$ the class of functions whose k-th derivatives extend continuously to each G_i. By $C_{k,\lambda}^0$ $(\overline{C}_{k,\lambda}^0)$ we shall denote the subclass of C_k^0 $(\overline{C}_k^0)$ consisting of functions whose k-th derivatives satisfy a Hölder condition; C_k $(\overline{C}_k)$ are functions whose k-th derivatives are continuous in G $(\overline{G})$. In describing the behavior of functions in a neighborhood of points of $\bigcup_{i,j=1}^{N} \gamma_{ij}$, we shall, without special mention, assume that the corresponding piece γ_{ij} is given in a local coordinate system by the equation $x_n = 0$.

2. For the elliptic equation the following is proved.

Theorem 1. Let a_{ij}, b_i, c, f belong to $\bar{C}_{0,\lambda}^0$ and be three times differentiable in a neighborhood of $\bigcup\limits_{i,j=1}^{N} \gamma_{ij}$ everywhere except at points belonging to $\bigcup \gamma_{ij}$, where these functions and their first derivatives may have jumps; let the surfaces γ_0, γ_{ij} be three times differentiable; let φ have two derivatives along γ_0 satisfying the Hölder condition.

Then the equation

$$\sum_{i,j=1}^{n} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} + cu = f \tag{2}$$

has a unique solution from $\bar{C}_{2,\lambda}^0 \cap \bar{C}_1$ with the boundary condition $u|_{\gamma_0} = \varphi$, satisfying (2) in each domain G_i. This solution can be obtained as the limit of solutions of equations of the form

$$\sum_{i,j} a_{ij}^h \frac{\partial^2 u^h}{\partial x_i \partial x_j} + b_i^h \frac{\partial u}{\partial x_i} + c^h u^h = f^h \tag{2'}$$

with the same boundary condition, where $a_{ij}^h, b_i^h, c^h, f^h$ are smooth functions converging to the coefficients and the right-hand side of (2).

The proof is carried out according to the following scheme. Uniqueness follows in an obvious way from the maximum principle, since, by continuity of the gradient u $(u \in \bar{C}_1)$, the maximum cannot be attained on γ_{ij}.

To prove existence, choose $a_{ij}^h, b_i^h, c^h, f^h$ satisfying the Hölder condition in $\bar{G}$, having in some neighborhood of $\bigcup\limits_{i,j=1}^{N} \gamma_{ij}$ derivatives, uniformly bounded with respect to h, up to the third order inclusive, not involving differentiation with respect to x_n, and such that the quadratic forms a^h_{ij} are uniformly nondegenerate with respect to h, while the functions

$$\Phi_h(x) = |a^h_{ij}(x) - a_{ij}(x)| + |b^h_i(x) - b_i(x)| +$$
$$+|c^h(x) - c(x)| + |f^h(x) - f(x)|$$

as $h \to 0$ tend to zero together with the Hölder constants uniformly in any closed subdomain of $\bigcup\limits_{i=1}^{N} G_i$. From Schauder estimates (3) it follows that the family u^h, $\partial u^h/\partial x_i$, $\partial^2 u^h/\partial x_i \partial x_j$, where u^h is the solution of (2′) with boundary condition φ, is bounded and satisfies the Hölder condition uniformly in h in any closed subdomain of $\bigcup\limits_{i=1}^{N} G_i$.

The smoothness of the coefficients and of the right-hand side of (2) in a neighborhood of $\bigcup\limits_{i,j=1}^{N} \gamma_{ij}$ makes it possible to give in this neighborhood an estimate, uniform in h, for the functions $(\partial u^h/\partial x_k)^2$, $(\partial^2 u^h/\partial x_k \partial x_l)^2$, $(\partial^3 u^h/\partial x_k \partial x_l \partial x_r)^2$, $k, l, r \neq n$. From it and from equation (2′) follow the uniform boundedness and equicontinuity of the family $\partial u^h/\partial x_n$.

Relying on these estimates, we can, by Arzelà' s theorem, choose a subsequence u^{h_k} with the following properties: 1) u^{h_k} and $\partial u^{h_k}/\partial x_i$ converge uniformly in $\bar{G}$; 2) $\partial^2 u^{h_k}/\partial x_i \partial x_j$ converge uniformly in each closed subdomain of $\bigcup\limits_{i=1}^{N} G_i$; 3) $\partial^2 u^{h_k}/\partial x_k \partial x_l$, for $k, l \neq n$, converge uniformly on $\bigcup\limits_{i,j=1}^{N} \gamma_{ij}$ and satisfy a Lipschitz condition uniformly in h. Passing to the limit, we obtain the limiting function u^0 from $C^0_2 \cap C^0_1$, and since in each G_i it satisfies equation (2) with smooth boundary conditions, it follows, by virtue of the restrictions on a_{ij}, b_i, c, f, that u^0 belongs to $\bar{C}^0_{2,\lambda}$.

Remark 1. From the proof of the theorem it is clear that $\partial^2 u/\partial x_k \partial x_l$ for $k,\ l \neq n$ are continuous in $\overline{G}$.

Remark 2. If the boundary function φ is only continuous, then u will belong to the class $C_1 \cap \overline{C}_0$.

Remark 3. From Theorem 1 one can derive the existence and uniqueness of the solution in the case of an unbounded domain when $C < -a,\ a > 0$, and $f(x) \to 0$ as $|x| \to \infty$, in the class of functions u tending to 0 as $|x| \to \infty$. The construction is carried out by means of a passage to the limit over an expanding sequence γ^i_0.

The case in which γ_{ij} intersect γ_0 is of interest. An additional investigation shows that the family $\partial u^h/\partial x_i$ will also be uniformly bounded in this case, but equicontinuity can be proved only outside a neighborhood of the boundary. Hence continuous differentiability follows everywhere in $\overline{G}$, except at the points of

$$\bigcup_{i,j=1}^{N} \gamma_0 \cap \gamma_{ij}.$$

3. For the parabolic equation the following holds:

Theorem 2. Let $G_T = G \times [0,T]$; suppose that the coefficients of the operator A do not depend on t and satisfy the restrictions of Theorem 1; $\varphi(x,t)$ is a smooth function on the base and lateral sides of the cylinder G_T. Suppose that $A^k f(x,0)$, $k = 0,1,2$; $A\dfrac{\partial f}{\partial t}(x,0)$; $\dfrac{\partial^2 f}{\partial t^2}$; $A^k \varphi(x,0)$, $k = 0,1,2,3$, are bounded and continuous.

Then the equation

$$\frac{\partial u}{\partial t} - \left(\sum_{ij} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + b_i \frac{\partial u}{\partial x_i} + cu \right) = f \tag{3}$$

has a unique solution u from $C^0_{2,\lambda} \cap \overline{C}_1$ in x, continuously differentiable with respect to t, and assuming on the base and lateral sides of G_T the values φ; this solution can be obtained as the limit of solutions of a sequence of equations of the form (3) with smoothed coefficients.

The proof is carried out by means of the construction described in item 2. The difference is that first one obtains an a priori estimate for $\partial u^h/\partial t$, $\partial^2 u^h/\partial t\, \partial x_k$, $\partial^3 u^h/\partial t\, \partial x_k\, \partial x_l$, $\partial^2 u^h/\partial t^2$, k, $l \neq n$. At this point one has to use the additional restrictions on f and φ.

To obtain the solution of the Cauchy problem one may either carry out a passage to the limit over an expanding sequence γ_0^i, or perform an independent construction, which is even somewhat simpler.

Theorem 3. Suppose that, for the coefficients of the operator A, the restrictions of Theorem 2 are satisfied uniformly in space, and that the manifolds γ_{ij} are at distances greater than a positive constant.

Then the Cauchy problem with initial function $\varphi(x)$, for which $A^k\varphi$, $k = 0,1,2,3$, are continuous and uniformly bounded in x, has for equation (3) a unique bounded solution u from $C^0_{2,\lambda} \cap \overline{C}_1$. This solution can be obtained as the limit of solutions of a sequence of equations of the form (3) with smoothed coefficients.

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