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Abstract

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ON RIEMANNIAN SPACES WITH A SUFFICIENTLY LARGE GROUP OF MOTIONS

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1°. As is known, spaces of constant curvature, and only they, possess the largest possible group of motions G_r for Riemannian spaces V_n, whose dimension is $r = n(n+1)/2$. G. Fubini proved (^{1}) that no space V_n can admit a full group of motions of dimension $r = n(n+1)/2 - 1$. Later it became clear that in the distribution of the dimensions of full groups of motions of spaces V_n there are entire gaps. From a number of works ($^{2-7}$) there follows the existence of at least two such gaps. Muto (^{8,9}) investigated groups of motions in non-conformally Euclidean spaces and found for them a third gap. However, the question of the existence of a third gap for all spaces V_n remained open, since conformally Euclidean spaces had not been investigated. The purpose of the present note is to fill this gap and thereby to give the final picture of the distribution of the dimensions of full groups of motions of Riemannian spaces within the limits of the three gaps indicated. At the same time, it gives an enumeration of all Riemannian spaces admitting a transitive or nontransitive group of motions of sufficiently large dimension ($r > (n-2)(n-3)/2 + 8$). As it turns out, all such spaces V_n for $n \neq 6$ and $n \neq 8$ are semi-reducible (^{10}). We note that the metric ds^2 of the space V_n is assumed positive-definite.

2°. We first formulate the theorems following from the works cited above, to which reference will be made below.

Theorem A (^{2,3}). A Riemannian space V_n, for $n \neq 4$, cannot admit a full group of motions G_r if

$$\frac{n(n-1)}{2} + 1 < r < \frac{n(n+1)}{2}$$

(the 1st gap).

Theorem B (^{4,7}). A Riemannian space V_n cannot admit a full group of motions G_r of dimension

$$\frac{(n-1)(n-2)}{2} + 5 < r < \frac{n(n-1)}{2}$$

(the 2nd gap).

Theorem C (^{4,5,7}). Every Riemannian space V_n admitting a group of motions of dimension $r > (n-1)(n-2)/2 + 5$ is a space of constant curvature or a subprojective space of Kagan.

Theorem D (^{8,9}). If a non-conformally Euclidean space V_n ($n > 6, \neq 8$) admits a group of motions of dimension $r > (n-2)(n-3)/2 + 8$, then necessarily $r = (n-1)(n-2)/2 + k$ ($k = 0, 1, 2, 3$), and the space is semi-reducible with a metric of the form

$$ds^2 = ds_0^2(x^1, x^2) + \sigma(x^1, x^2)\, ds_1^2(x^3, \dots, x^n),$$

where ds_1^2 is an $(n-2)$-dimensional metric of constant curvature.

In addition, we shall need the following theorem:

Theorem E (^{11}). The full orthogonal group $O(n)$, for $n \neq 4$, cannot contain subgroups whose dimension is greater than the dimension of $O(n-1)$.

3°. Consider a Riemannian space V_n admitting a group of motions G_r. Denote by H_m the stationary subgroup of some point of the space, and by $\widetilde{H}_m$ the isotropy group, i.e. the group of rotations in the tangent space E_n of this point induced by the group H_m. In a joint paper of the author and Gu Chao-hao the following theorem was proved:

Theorem F (^{12}). Let the group of motions G_r be transitive, and let the isotropy group $\widetilde{H}_m$ decompose into the direct product of two subgroups $H^{(0)}$ and $H^{(1)}$, acting on mutually orthogonal complementary planes E_{n-q} and E_q. If $H^{(1)}$ is irreducible in E_q and admits no one-dimensional groups of rotations in E_q permutable with it, then the space V_n is semireducible. Its metric can be brought to the form

$$ds^2 = ds_0^2(x^1, \dots, x^{n-q}) + \sigma(x^1, \dots, x^{n-q}) ds_1^2(x^{n-q+1}, \dots, x^n), \tag{1}$$

and the group G_r is a non-mixing group of motions with respect to the semireducible decomposition (1).

The following supplements may be made to this theorem:

1) If $\sigma \neq$ const in (1), then one can show that the metric ds_1^2 is Euclidean, and the decomposition (1) reduces to the form

$$ds^2 = ds_0^2(x^1, \dots, x^{n-q}) + e^{2ax^1}(dx^{n-q+1\,2} + \dots + dx^{n\,2}). \tag{2}$$

2) The requirement that $H^{(1)}$ admit no one-dimensional groups G_1 of rotations in E_q permutable with it is fulfilled automatically if q is odd. If q is even, then this requirement is always fulfilled when the dimension of the group $H^{(1)}$ is greater than $(q/2)^2$. Indeed, as É. Cartan showed, the

infinitesimal matrix of the group of rotations G_1, permutable with an irreducible group of orthogonal matrices, is brought to the form $I = \begin{Vmatrix} 0 & E \\ -E & 0 \end{Vmatrix}$, where E is the identity matrix of order $q/2$. Every skew-symmetric matrix permutable with I, as is easily verified, has the form $L = \begin{Vmatrix} A & B \\ -B & A \end{Vmatrix}$, where A is a skew-symmetric and B a symmetric matrix of order $q/2$. The maximal number of parameters which the matrix L can contain is

$$\tfrac{1}{2}[q/2(q/2-1)] + \tfrac{1}{2}[q/2(q/2+1)] = (q/2)^2.$$

4°. **Lemma 1.** If the isotropy group $\widetilde{H}_m$ leaves invariant a q-dimensional plane E_q, then

$$r < \frac{n(n+1)}{2} - q(n-q).$$

Indeed, the orthogonal complement E_{n-q} to E_q is also invariant with respect to $\widetilde{H}_m$, and therefore its dimension m cannot exceed the sum of the dimensions of the full rotation groups in E_q and E_{n-q}. Hence the required estimate follows at once, since $r \leq m + n$.

From Lemma 1 the following assertions follow in an obvious way:

Lemma 2. If $r > (n-2)(n-3)/2 + 6$, then $\widetilde{H}_m$ cannot fix any plane E_q for $3 \leq q \leq n-3$.

Lemma 3. Let ρ_i be the roots of the Ricci tensor R_{ij} of the space V_n, admitting a group of motions G_r of dimension $r > (n-2)(n-3)/2 + 6$. Then only three cases are possible: 1) $\rho_1 = \rho_2 = \ldots = \rho_n$; 2) $\rho_1 \neq \rho_2 = \ldots = \rho_n$; 3) $\rho_1, \rho_2 \neq \rho_3 = \ldots = \rho_n$.

5°. We shall investigate a Riemannian space V_n with a transitive group of motions G_r of dimension $r > (n-2)(n-3)/2+6$, assuming that $\widetilde{H}_m$ is reducible. According to Lemma 2 there are only the following possibilities:

I. $E_n = E_1 \times E_{n-1}$ and $\widetilde{H}_m$ is an irreducible group of rotations in E_{n-1}.

II. $E_n = E_2 \times E_{n-2}$, and either $\widetilde{H}_m$ itself or its subgroup $\widetilde{H}_{m-1}$ is an irreducible group of rotations in E_{n-2}. In this case the dimension of the stationary group satisfies the inequality $m > (n-3)(n-4)/2 + 3$.

In case I all the conditions of Theorem F are fulfilled. The absence of interchange rotations from $\widetilde{H}_m$ on E_{n-1} follows from the fact that for $n > 3$ the dimension of $\widetilde{H}_m$ is greater than $((n-1)/2)^2$. Therefore

$$ds^2 = dx^{1\,2} + \sigma(x^1)\, ds_1^2(x^2, \ldots, x^n). \tag{3}$$

The group G_r is a non-interchanging transitive group of motions in (3). Therefore, for $\sigma = \text{const}$, G_r decomposes into the direct product of the group of translations along the lines x^1 and the subgroup G_{r-1} of motions in ds_1^2. Since

$$r - 1 > (n-2)(n-3)/2 + 5,$$

it follows, by Theorem C, that the space ds_1^2 must have constant curvature K, since in a subprojective space the isotropy group is reducible. Consequently, every space (3) is a subprojectively reducible space for $K \neq 0$, or Euclidean for $K = 0$. If $\sigma \neq \text{const}$, then one obtains the metric

$$ds^2 = ds^{1\,2} + e^{2ax^1}(dx^{2\,2} + \dots + dx^{n\,2}),$$

which has constant negative curvature.

Case II is possible only for $n > 6$, since for $n \leqslant 6$ the dimension of $\widetilde{H}_m$ is greater than the dimension of $O(n-2)$. If $n > 6$, then, by dimension, $\widetilde{H}_m$ occupies an intermediate position between $O(n-2)$ and $O(n-3)$, and therefore, by Theorem E, the group $\widetilde{H}_m$ or its subgroup $\widetilde{H}_{m-1}$ coincides with $O(n-2)$. Consequently,

$$m = (n-2)(n-3)/2$$

or

$$m = (n-2)(n-3)/2 + 1,$$

and

$$r = (n-1)(n-2)/2 + 2$$

or

$$r = (n-1)(n-2)/2 + 3.$$

In both cases the conditions of Theorem F are fulfilled and

$$ds^2 = ds_0^2(x^1, x^2) + \sigma(x^1, x^2)\, ds_1^2(x^3, \dots, x^n), \tag{4}$$

where ds_1^2 must necessarily have constant curvature K_1. Since the group G_r is non-interchanging, ds_0^2 admits a group of motions G_2 or G_3, and therefore also has constant curvature K_0. For $\sigma = \text{const}$, the space V_n thus decomposes into the direct product of two-dimensional and $(n-2)$-dimensional spaces of constant curvature, and then

$$r = (n-1)(n-2)/2 + 3.$$

For $\sigma \neq \text{const}$ it is easy to find that the metric reduces to the form

$$ds^2 = dx^{1\,2} + e^{2bx^1} dx^{2\,2} + e^{2ax^1}(dx^{3\,2} + \dots + dx^{n\,2}), \qquad a \neq 0, \quad b \neq 0, \quad a \neq b. \tag{5}$$

The group of motions in this case has dimension

$$r = (n-1)(n-2)/2 + 2.$$

6°. If the space V_n is not Einstein, then among the roots of the Ricci tensor there are at least two distinct ones; and since the corresponding eigenspaces in E_n are invariant with respect to the group of motions, $\widetilde{H}_m$ is reducible. In this case, from the arguments of item 5° there follows the theorem:

Theorem 1. *If a non-Einstein n-space V_n admits a transitive group of motions G_r of dimension*

$$r > (n-2)(n-3)/2 + 6,$$

then this space belongs to one of the following types:

1) V_n *of constant curvature,* $r = n(n+1)/2$.
2) $V_n = V_1 \times V_{n-1}$, *where* V_{n-1} *is of constant curvature* $K \neq 0$, $r = n(n-1)/2 + 1$.
3) $V_n = V_2 \times V_{n-2}$, *where* V_2 *and* V_{n-2} *are of constant curvature,*

$$r = (n-1) \times (n-2)/2 + 3.$$

4) V_n *is defined by the metric* (5),

$$r = (n-1)(n-2)/2 + 2.$$

An Einstein space of nonconstant curvature is not conformally Euclidean. Assuming

$$r > (n-2)(n-3)/2 + 8 \quad (n > 6,\ \neq 8)$$

and applying Theorem D to it, we arrive at case 3) of Theorem 1. For $n = 6$ and $n = 8$ the question is still not completely clarified. It is known that in these cases, to the types of spaces listed in Theorem 1, there are added the symmetric V_6 and V_8 belonging to the class of spaces V_{2p} with group of motions G_{p^2+2p}, having an irreducible isotropy group $\widetilde{H}_{p^2}$ ([18]). As a result one obtains:

Theorem 2. Every Riemannian space V_n admitting a transitive group of motions of dimension $r > (n-2)(n-3)/2 + 8$ $(n > 6, \neq 8)$ belongs to one of the types indicated in Theorem 1.

We note that, using Lemma 3 and the condition of constancy of the Ricci roots ρ_i, which follows from the transitivity of G_r, one can prove Theorem 1 specifically for conformally Euclidean spaces (without the Einstein condition). In this case case 4) drops out, since it gives a non-conformally Euclidean space.

7°. Let us now turn to spaces V_n with a nontransitive group of motions G_r of dimension $r > (n-2)(n-3)/2 + 6$ $(n > 4)$. In this case the orbits of the group may be:

I. Geodesically parallel hypersurfaces V_{n-1} $(n > 4)$.
II. Surfaces V_{n-2} $(n > 8)$.

In case I, if the isotropy group $\widetilde{H}_m$ is irreducible in E_{n-1}, the tangent space to V_{n-1}, then it can be shown that the space V_n is subprojective. Its metric has the form (3), where ds_1^2 is a metric of constant curvature, and $\sigma \neq \text{const}$. If the group $\widetilde{H}_m$ is reducible in E_{n-1}, then as a result we obtain

$$ds^2 = dx^{1^2} + \varphi(x^1)\, dx^{2^2} + \sigma(x^1)\, ds_1^2(x^3, \dots, x^n), \tag{6}$$

where ds_1^2 is a metric of constant curvature, and the dimension of the group G_r is $(n-1)(n-2)/2 + 1$.

In case II the group $\widetilde{H}_m$ is irreducible in E_{n-2}, the tangent space to the orbit V_{n-2}, and as a result we obtain the metric (4), where ds_1^2 has constant curvature, and $r = (n-1)(n-2)/2$.

Thus we obtain the theorem:

Theorem 3. If the space V_n admits a nontransitive group of motions G_r of dimension $r > (n-2)(n-3)/2+6$ $(n > 4)$, then it belongs to one of the following types:

1) V_n is subprojective, $r = n(n-1)/2$, the orbits of the group are V_{n-1}.
2) V_n is defined by the metric (6), where ds_1^2 has constant curvature, $r = (n-1)(n-2)/2 + 1$, the orbits of the group are V_{n-1}.
3) V_n is defined by the metric (4), where ds_1^2 has constant curvature, $r = (n-1)(n-2)/2$, the orbits of the group are V_{n-2}.

8°. From the preceding results the following conclusions also follow:

Theorem 4. Every Riemannian space admitting a transitive or nontransitive group of motions G_r of dimension $r > (n-2)(n-3)/2 + 8$ $(n \neq 6, 8)$ is a semireducible space.

Theorem 5. A Riemannian space V_n cannot admit a complete group of motions G_r whose dimension satisfies one of the inequalities:

$$1) \qquad \frac{n(n-1)}{2} + 1 < r < \frac{n(n+1)}{2} \qquad (n \neq 4);$$

$$2) \qquad \frac{(n-1)(n-2)}{2} + 3 < r < \frac{n(n-1)}{2} \qquad (n \neq 6, 8);$$

$$3) \qquad \frac{(n-2)(n-3)}{2} + 8 < r < \frac{(n-1)(n-2)}{2}.$$

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