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Abstract

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ON THE EXISTENCE OF NONRIGID CLOSED SURFACES

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1. In 1929, S. Cohn-Vossen proved the existence of nonrigid closed surfaces having at least one nontrivial infinitesimal bending, and also posed the question of the existence of a closed surface with several, or even with a countable number of, linearly independent infinitesimal bendings ([2]). In the works of S. Cohn-Vossen ([2]) and E. G. Poznyak ([2]), this question is connected with the existence of infinitesimal bendings of the second order, and in the work of N. V. Efimov ([3]), with the analytic unbendability of a closed surface. Thus, the question of the existence of a closed surface with several linearly independent infinitesimal bendings arises naturally in the theory of infinitesimal bendings.

In 1957, E. G. Poznyak constructed an example of a closed surface of revolution with a singularity at one pole, whose fundamental system of infinitesimal bendings contains a countable set of bendings ([4]). In 1958, the author constructed an example of a system of nonrigid smooth closed surfaces whose fundamental system of infinitesimal bendings contains at least two bendings ([5]). In the present note an example is constructed that proves the existence of nonrigid regular closed surfaces having no fewer than two linearly independent infinitesimal bendings.

2. The finding of infinitesimal bendings of a surface of revolution defined by the equation of the meridian $r = r(u)$ reduces ([1]) to finding such nontrivial integrals $\chi_k(u)$ of the equation

$$r\chi_k'' + r''\chi_k(k^2 - 1) = 0, \tag{1}$$

which are continuous together with the first derivative everywhere on the surface and vanish at its poles. Here k may take any integer values $k \geq 2$, and different values of k correspond to linearly independent infinitesimal bendings.

Let the surface of revolution S be given by the following equations of the meridian:

$$r = \begin{cases} 0, & \text{for } u = 0, \\ \sqrt{uK_0(bu^2)}, & \text{for } 0 < u \le a, \\ A\sqrt{(u-a-c)^2+1}\ \mathrm{ch}(\alpha\ \mathrm{arctg}(u-a-c)), & \text{for } a < u \le a+c, \\ r(2a+2c-u), & \text{for } a+c \le u \le 2a+2c, \end{cases} \tag{2}$$

where a, b, c, A, α are as yet undetermined constants, and $K_0(bu^2)$ is the modified Bessel function of the second kind (Macdonald function).

For $\chi_k(u)$ with $k = 2$ and with $k = 6$, in view of (1) one may set

$$\chi_2 = \begin{cases} u^{-1/2}\sin b\sqrt{3u^2}, & \text{for } u \leqslant a, \\ B_2\sqrt{(u-a-c)^2+1}\cos(\sqrt{4+3\alpha^2}\ \mathrm{arctg}(u-a-c)), & \text{for } a < u \leqslant a+c, \\ \chi_2(2a+2c-u), & \text{for } a+c \leqslant u \leqslant 2a+2c; \end{cases}$$

$$\chi_6 = \begin{cases} u^{-5/2}(\sin b\sqrt{35u^2} - b\sqrt{35}\,u^2\cos b\sqrt{35u^2}), & \text{for } u \leqslant a, \\ B_6\sqrt{(u-a-c)^2+1}\cos(\sqrt{36+35\alpha^2}\ \mathrm{arctg}(u-a-c)), & \text{for } a < u \leqslant a+c, \\ \chi_6(2a+2c-u), & \text{for } a+c \leqslant u \leqslant 2a+2c, \end{cases}$$

where B_2 and B_6 are as yet undetermined constants.

The surface S is regular everywhere except, possibly, for the parallel $u = a$. The integrals $\chi_2(u)$ and $\chi_6(u)$ vanish in the belts of the surface S and are continuous, together with their first and second derivatives, everywhere on the surface except, possibly, for the parallel $u = a$. In order that the functions $r(u)$, $\chi_2(u)$, and $\chi_6(u)$ be twice continuously differentiable also at $u = a$, the constants $a, b, c, A, \alpha, B_2, B_6$ must satisfy the conditions:

$$a^{1/2}K_0(ba^2) = A(c^2+1)^{1/2}(\mathrm{ch}\,\alpha\ \mathrm{arctg}\,c);$$

$$\frac{1}{2}a^{-1/2}K_0(ba^2) - 2ba^{3/2}K_1(ba^2) =$$

$$= A(c^2+1)^{-1/2}[-c\ \mathrm{ch}(\alpha\ \mathrm{arctg}\,c) - \alpha\ \mathrm{sh}(\alpha\ \mathrm{arctg}\,c)];$$

$$\left(-\frac{1}{4}a^{-2} + 4b^2a^2\right)a^{1/2}K_0(ba^2) = A(1+\alpha^2)(1+c^2)^{-3/2}\,\mathrm{ch}(\alpha\ \mathrm{arctg}\,c);$$

$$a^{-1/2} + \sin\sqrt{3ba^2} = B_2(c^2+1)^{1/2}\cos(\sqrt{4+3\alpha^2}\ \mathrm{arctg}\,c);$$

$$-\frac{1}{2}a^{-3/2}\sin\sqrt{3ba^2}+2b\sqrt{3}\,a^{1/2}\cos\sqrt{3ba^2} \tag{3}$$

$$=-B_2(c^2+1)^{-1/2}[c\cos(\sqrt{4+3\alpha^2}\ \mathrm{arctg}\,c)-\sqrt{4+3\alpha^2}\sin(\sqrt{4+3\alpha^2}\ \mathrm{arctg}\,c)];$$

$$a^{-5/2}(\sin\sqrt{35ba^2}-\sqrt{35}\,ba^2\cos\sqrt{35ba^2})=B_6(c^2+1)^{1/2}\cos(\sqrt{36+35\alpha^2}\ \mathrm{arctg}\,c);$$

$$-\frac{5}{2}a^{-7/2}(\sin\sqrt{35ba^2}-\sqrt{35ba^2}\cos\sqrt{35ba^2})+70b^2a^{1/2}\sin\sqrt{35ba^2}=$$

$$=-B_6(c^2+1)^{-1/2}[c\cos(\sqrt{36+35\alpha^2}\ \mathrm{arctg}\,c)-\sqrt{36+35\alpha^2}\sin(\sqrt{36+35\alpha^2}\ \mathrm{arctg}\,c)].$$

If, as the constants, one takes solutions of the system (3) satisfying the geometrically obvious conditions

$$a>0,\qquad c>0,\qquad ba^2>\frac{1}{4} \tag{4}$$

(the last condition ensures the nonconvexity of the surface S), then the surface S, given by equations (2), will be a nonrigid regular closed surface having at least two linearly independent infinitesimal bendings.

3. The existence of solutions of the system (3) satisfying conditions (4) is proved as follows. Eliminating from the system (3) the unknowns A, B_2, B_6, and making the change of variables $ba^2=d$, $2a/(1+c^2)=e$,

we obtain a system of 4 equations with 4 unknowns:

$$\begin{gathered}1-4\sqrt{3}\,d\ \mathrm{ctg}\,\sqrt{3}d+e\left[\sqrt{4+3\alpha^2}\ \mathrm{tg}\left(\sqrt{4+3\alpha^2}\,\mathrm{arctg}\,c\right)-c\right]=0;\\ 5+\frac{1}{\sqrt{35}\,d\ \mathrm{ctg}\,\sqrt{35}d-1}+e\left[\sqrt{36+35\alpha^2}\ \mathrm{tg}\left(\sqrt{36+35\alpha^2}\,\mathrm{arctg}\,c\right)-c\right]=0;\\ 1-16d^2+(1+\alpha^2)e^2=0;\qquad 1-4d\frac{K_1(d)}{K_0(d)}+e[\alpha\ \mathrm{th}(\alpha\ \mathrm{arctg}\,c)+c]=0.\end{gathered} \tag{5}$$

System (5) has the following approximate solution:

$$d=1.676476182;\qquad \alpha=0.676302299;\qquad c=1.032224126;\qquad e=5.492713640. \tag{6}$$

This approximation satisfies the conditions of L. V. Kantorovich' s theorem on the existence of an exact solution of a system of algebraic and transcendental equations

$$f_i(x_1, x_2, \dots, x_m) = 0 \qquad (i = 1, 2, \dots, m) \tag{7}$$

in a neighborhood of the approximate solution $(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)})$.

Let, for the system of equations (7):

1) $|f_i(x_1^{(0)}, x_2^{(0)}, \dots, x_m^{(0)})| \le \eta$ $(i = 1, 2, \dots, m)$;
2) the matrix $\|(\partial f_i/\partial x_k)_0\|$ $(i, k = 1, 2, \dots, m)$ has determinant Δ different from zero; if A_{ik} are its minors, then put

$$\max_i \frac{1}{|\Delta|} \sum_k |A_{ik}| = B;$$

3) $|\partial^2 f_i/\partial x_j \partial x_k| \le L$ in the region of interest to us.

Then, if

4) $h = B^2 m^2 \eta L \le 1/2$, then a solution of system (7) exists and can be obtained by Newton' s method [6].

The region referred to in condition 3) is determined by the inequalities

$$|x_i^{(0)} - x_i| \le \frac{1 - \sqrt{1 - 2h}}{h} B\eta.$$

For the approximate solution (6)

$$\eta < 10^{-7}; \qquad B^2 < 0.15; \qquad L < 6000; \qquad m = 4; \qquad h < 0.0015 < 1/2,$$

therefore system (5), and together with it system (3), have an exact solution satisfying conditions (4).

The approximate values of the constants are as follows:

$$a = 5.672563337\dots; \qquad b = 0.0521001148\dots; \qquad c = 1.032224126\dots;$$

$$\alpha = 0.676302299\dots;$$

$$A = 0.245606900\dots; \qquad B_2 = -0.250766854\dots; \qquad B_6 = -0.0414451514\dots.$$

All the signs written out are correct.

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CITED LITERATURE

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[6] L. V. Kantorovich, *DAN*, **59**, No. 7, 1237 (1948).

Note: Figure translations are in progress. See original paper for figures.