MATHEMATICS

1960

SovietRxiv

Abstract

Full Text

MATHEMATICS

R. A. ALEKSANDRYAN

DISCONTINUOUS SOLUTIONS OF THE FIRST HOMOGENEOUS BOUNDARY-VALUE PROBLEM FOR THE EQUATION OF VIBRATION OF A STRING

(Presented by Academician S. L. Sobolev on 11 V 1960)

1. In the work (^{1}) it was shown that the investigation of the spectral properties of operators generated by one class of systems of differential equations of S. L. Sobolev type can be reduced to the study of the following boundary-value problem for eigenvalues:

$$Mu + \lambda Lu = 0, \tag{1}$$

$$u\big|_{\Gamma} = 0, \tag{2}$$

where M and L are formally self-adjoint differential operators of the second order with an arbitrary number of independent variables, and the quadratic form corresponding to the operator L is uniformly definite in the domain D under consideration, with boundary Γ.

In this note we shall adhere to the definitions and notation introduced in the author’ s work (^{1}).

2. Let Φ be the linear topological space of functions infinitely differentiable in $D + \Gamma$, where the linear operations are the usual ones, and convergence in the space Φ means uniform convergence, together with the derivatives of all orders, in the closed domain $\overline{D}$.

In the theory of boundary-value problems for elliptic equations it is known that if the coefficients of the operators M and L belong to Φ, and the boundary Γ is arbitrarily smooth, then $Q\Phi \subseteq \Phi_0$, and from $\varphi_n(x) \xrightarrow{\Phi} 0$ it follows that $Q\varphi_n(x) \xrightarrow{\Phi} 0$.

We introduce the following important definition:

Definition. A nonzero functional $T_\lambda \in \Phi^*$ is called a **proper functional** of the operator Q, or of the boundary-value problem (1), (2), corresponding to the eigenvalue λ, if for all $\varphi(x) \in \Phi_0$ the relation

$$T_\lambda(M\varphi + \lambda L\varphi) = 0, \qquad \varphi(x) \in \Phi_0 \tag{3}$$

holds.

Let the matrix $C(x) = A(x) \cdot B(x) = \|c_{i,j}(x)\|$, and let γ_λ be the set of points on the boundary Γ at which

$$\sum_{i,j=1}^{n} (c_{i,j} + \lambda b_{i,j}) \cos nx_i \cos nx_j = 0.$$

Lemma 1. If the proper functional $T_\lambda(\varphi)$ is isomorphic to a function $U_\lambda(x)$ smooth in the closed domain $\overline{D}$, and if the complement of the set γ_λ is everywhere dense on Γ, then $u_\lambda(x)$ is a nontrivial solution of the boundary-value problem (1), (2) in the classical sense.

3. We shall investigate in detail the problem (1), (2) in the simplest case, when $Mu = u_{xx} - u_{yy}$, $Lu = u_{xx} + u_{yy}$; then equation (1), for

for $|\lambda| < 1$ becomes the classical equation of vibration of a string

$$(1+\lambda)\frac{\partial^2 u}{\partial x^2} - (1-\lambda)\frac{\partial^2 u}{\partial y^2} = 0. \tag{1*}$$

We shall agree to regard the domain D as "admissible," i.e., such that any straight line intersects its boundary Γ, assumed to be sufficiently smooth and homeomorphic to a circle, in no more than two points.

Let λ belong to the interval $(-1, +1)$, and let $y - \mu x = \text{const}$ (respectively $y + \mu x = \text{const}$) be the first (respectively second) family of real characteristics of equation (1*), where

$$\mu^2 = \frac{1-\lambda}{1+\lambda}.$$

We define the automorphism $S_\lambda^{(+)}$ of the boundary Γ as follows: to each point $\theta \in \Gamma$ we assign the point of intersection with the boundary Γ of the characteristic of the first family passing through θ. In a completely analogous way, replacing only the characteristics of the first family by the characteristics of the second family, we define the automorphism $S_\lambda^{(-)}$. Finally, define the automorphism S_λ as the product of the automorphisms $S_\lambda^{(+)}$ and $S_\lambda^{(-)}$, putting $S_\lambda = S_\lambda^{(-)} \cdot S_\lambda^{(+)}$.

Let $A_r(\lambda, \Gamma)$ be the set of points $\theta \in \Gamma$ that are fixed under the r-th iteration of the automorphism S_λ, or periodic with period r. It is quite obvious that

$$\Gamma = A_\infty(\lambda, \Gamma) + \sum_{r=1}^{\infty} A_r(\lambda, \Gamma),$$

where $A_\infty(\lambda, \Gamma)$ is the set of so-called aperiodic points, i.e., those for which $S_\lambda^r \theta \neq \theta$ for any r. It turns out that all periodic points of the automorphism S_λ have one and the same period $r = r(\lambda, \Gamma)$, so that in the most general case $\Gamma = A_\infty(\lambda, \Gamma) + A(\lambda, \Gamma)$, and the set of all periodic points $A(\lambda, \Gamma)$ is closed.

It is further obvious that the family of automorphisms S_λ is also a family of special topological mappings of the boundary Γ onto itself, which preserve orientation.

It turns out that the study of the first homogeneous boundary-value problem (1*), (2) essentially reduces to the study of the properties of the family of automorphisms S_λ, depending on the real parameter λ.

The countable set of points of the form $S_\lambda^k \theta$ $(k = 0, \pm 1, \pm 2, ...)$ is called the **trajectory** of the automorphism S_λ, **generated by the point** θ. The trajectories generated by the points $S_\lambda^{(+)}\theta$ and $S_\lambda^{(-)}\theta$ shall be called **adjacent** to the one generated by the point θ; then it is clear that adjacent trajectories coincide, and the notion of adjacency of trajectories is a mutual one.

Each point $\theta \in \Gamma$ generates a completely definite set of boundary points $\mathfrak{M}(\lambda, \theta, \Gamma)$, which is the union of two adjacent trajectories generated by this point, and is called the λ-cycle generated by the point θ. We shall call a λ-cycle **trivial** if it contains a point at which the tangent to the boundary Γ is parallel to one of the characteristic directions.

Obviously, $\mathfrak{M}(\lambda, \theta, \Gamma)$ consists of a finite number of distinct points if and only if θ is a periodic point of the automorphism S_λ; in this case the λ-cycle will be called **closed**.

It is known that any solution of equation (1^*) can be represented in the form

$$u(x, y, \lambda) = f_+(y - \mu x) + f_-(y - \mu x). \tag{4}$$

Lemma 2. *Let $u(x, y, \lambda_0)$ be representable in the form (4), where f_+, f_- are piecewise-continuous functions, and suppose there exists $\theta_0 \in \Gamma$ such that $\mathfrak{M}(\lambda_0, \theta_0, \Gamma)$ is everywhere dense on the boundary Γ. Then, if $u(x, y, \lambda_0)$ satisfies condition (2), it is identically equal to zero.*

Theorem 1. Let $u(x, y, \lambda_0)$ have the form (4), where f_+, f_- are arbitrary measurable functions, and satisfy the boundary condition (2). Then, if the boundary of the domain has bounded curvature and the set of periodic points $A(\lambda_0, \Gamma)$ is empty, then $u(x, y, \lambda_0)$ is equal to zero almost everywhere.

Remark. Under the hypotheses of Lemma 2, one cannot, in general, prescribe a priori the values of any function f_+ or f_- even at two points, and still less on some arc.

Lemma 3. Let an admissible domain D and a natural number $N \geq 2$ be given. Then there exist a point $\theta_0 \in \Gamma$ and a value of the parameter $\lambda = \lambda_0$ such that $\mathfrak{M}(\lambda_0, \theta_0, \Gamma)$ is a nontrivial closed λ-cycle consisting of $2N$ points.

Lemma 4. Let D be an arbitrary admissible domain. The set of those values of the parameter λ for which the set $A(\lambda, \Gamma)$ is nonempty is everywhere dense on the interval $(-1, +1)$.

This lemma is of special interest from the point of view of the boundary-value problem $(1^*), (2)$, because the following fundamental theorem holds:

Theorem 2. Let D be an arbitrary admissible domain, and suppose that for $\lambda = \lambda_0$ some iteration of the automorphism S_λ has a fixed point θ_0 such that the λ-cycle $\mathfrak{M}(\lambda_0, \theta_0, \Gamma)$ is not trivial. Then there exists a proper functional of the boundary-value problem $(1^*), (2)$, isomorphic to a bounded function.

Assuming that the hypotheses of the theorem are fulfilled, one can construct a bounded function $u(x, y, \lambda_0, \theta_0)$, not identically zero, such that

$$\iint_D u(x, y, \lambda_0, \theta_0) \left\{ (1+\lambda_0) \frac{\partial^2 \varphi}{\partial x^2} - (1-\lambda_0) \frac{\partial^2 \varphi}{\partial y^2} \right\} dx\, dy = 0, \qquad \varphi(x, y) \in \Phi_0. \tag{5}$$

Let r be the period, i.e. $S^r_{\lambda_0} \theta_0 = \theta_0$, whereas for all $p < r$, $S^p_{\lambda_0} \theta_0 \neq \theta_0$. Through the points $S^k_{\lambda_0} \theta_0$ and $S^k_{\lambda_0} S^{(+)}_{\lambda_0} \theta_0$ $(k = 0, 1, 2, ...)$, draw the characteristics of the first family of equation (1^*). It is clear that there will be only r distinct characteristics, which divide the whole plane into $r - 1$ strips and two half-strips. Define on the whole plane a piecewise constant function $u_1(x, y, \lambda_0, \theta_0)$ as follows: in the right half-strip put $u_1 = -1$, in the strip adjacent to it put $u_1 = +1$, in the next strip $u_1 = -1$, and so on. Then $u_1(x, y, \lambda_0, \theta_0)$ will be defined everywhere. Now draw, from the same points $S^k_{\lambda_0} \theta_0$ and $S^k_{\lambda_0} S^{(+)}_{\lambda_0} \theta_0$ $(k = 0, 1, 2, ...)$, the characteristics of the second family and, proceeding in exactly the same way, construct on the whole plane a piecewise constant function $u_2(x, y, \lambda_0, \theta_0)$. Observe that if, from the above-mentioned $2r$ points, characteristics of both families are drawn, then they will divide the domain D into a finite number of parallelograms $D_1, D_2, ...$ and curvilinear triangles (four of them will be digons).

Now construct the function

$$u(x, y, \lambda_0, \theta_0) = u_1(x, y, \lambda_0, \theta_0) \pm u_2(x, y, \lambda_0, \theta_0),$$

where the sign must be chosen so that the function $u(x, y, \lambda_0, \theta_0)$ is equal to zero in one of the digons.

Imagine the domain D colored in two colors according to the following rule: partial domains whose boundaries have only one common point are colored in the same color, while if their boundaries have a common segment of a characteristic, they are colored in different colors. For definiteness suppose that one of the curvilinear triangles (and therefore all the others) is colored white; then, under the stated rule, all partial domains will be colored in a completely definite color.

In the parts of the domain D colored white, the function $u(x, y, \lambda_0, \theta_0)$, obviously, is equal to zero, while in the black parallelograms it is equal either to $+2$ or to -2, in such a way, however, that in parallelograms with a common vertex it assumes different values.

It is now obvious that

$$\iint\limits_D u(x, y, \lambda_0, \theta_0) \left\{ (1+\lambda_0)\frac{\partial^2\varphi}{\partial x^2} - (1-\lambda_0)\frac{\partial^2\varphi}{\partial y^2} \right\} dx\, dy = \sum \iint\limits_{D_p} \pm 2 \left\{ (1+\lambda_0)\frac{\partial^2\varphi}{\partial x^2} - (1-\lambda_0)\frac{\partial^2\varphi}{\partial y^2} \right\} dx\, dy, \tag{6}$$

where the summation extends only over the black parallelograms, and the signs are taken in the indicated manner.

The integrals over the individual parallelograms are computed elementarily, if one takes into account that the boundary D_p consists of pieces of characteristics. Indeed:

$$\iint\limits_{D_p} \left\{ (1+\lambda_0)\frac{\partial^2\varphi}{\partial x^2} - (1-\lambda_0)\frac{\partial^2\varphi}{\partial y^2} \right\} dx\, dy = \{\varphi(b_p) + \varphi(d_p) - \varphi(a_p) - \varphi(c_p)\}, \tag{7}$$

where a_p, b_p, c_p, d_p are the vertices of the parallelogram D. Substituting (7) into (6), we are convinced of the validity of Theorem 1, since in the resulting sum the value of the function $\varphi(x, y)$ at vertices common to two parallelograms occurs twice and with different signs, while the remaining vertices belong to the boundary, where $\varphi(x, y)$ is equal to zero by condition.

Theorem 3. *Let D be an arbitrary admissible domain, and suppose that for $\lambda = \lambda_0$ and $\theta_0 \in \Gamma$, $\mathfrak{M}(\lambda_0, \theta_0, \Gamma)$ is a nontrivial closed λ-cycle. Then the point λ_0 belongs to the spectrum of the operator Q, and consequently also to the spectrum of the operator $\mathfrak{A}$.*

Theorems 2 and 3 give sufficient conditions, in terms of the automorphisms introduced above, under which there exists an eigenfunctional, or the point $\lambda = \lambda_0$ belongs to the spectrum. It turns out that in these same terms one can indicate a sufficient condition for the existence of a smooth eigenfunction.

Theorem 4. *If the set $A(\lambda_0, \Gamma)$ has at least one interior point, then λ_0 is an eigenvalue of the operator Q, and consequently also of the operator $\mathfrak{A}$.*

In the case when the domain D is a disk, there is a complete system of smooth eigenfunctions, and accordingly the spectra of the operators under consideration turn out to be purely point spectra.

In conclusion I express my deep gratitude to Academician S. L. Sobolev for his attention to the present work and for valuable remarks.

Yerevan State University

Received
17 XI 1959

REFERENCES

1. R. A. Aleksandryan, DAN, 131, No. 3 (1960).
2. S. L. Sobolev, Izv. AN SSSR, ser. matem., 18, 3 (1954).
3. R. A. Aleksandryan, Dissertation, Moscow State University, 1949.
4. R. A. Aleksandryan, DAN, 73, No. 5 (1960).

Note: Figure translations are in progress. See original paper for figures.