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Abstract

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ON THE SOLUTION OF THE INTEGRAL EQUATION OF COASTAL REFRACTION OF ELECTROMAGNETIC WAVES

(Presented by Academician V. A. Ambartsumian, 3 V 1960)

In the theory of coastal refraction of electromagnetic waves (1) one encounters the equation

$$B(\tau) = \frac{\alpha}{\pi} \int_0^\infty K_0\left(|\tau - \tau'|\right) B(\tau')\, d\tau' + g(\tau), \tag{1}$$

where $K_0(\tau)$ is the Macdonald function; $g(\tau)$ is a function characterizing the wave incident on the shore. V. A. Fok (1,2) found the solution of equation (1) for the case $g(\tau) = e^{-\tau}$. This case corresponds to the incidence on the shore of a plane wave.

Recently V. V. Sobolev (3), generalizing the works of V. A. Ambartsumian (4) on the solution of integral equations of the theory of radiative transfer, as well as his own works in this field (5), developed a new method for solving equations with kernels depending on the modulus of the difference of two arguments, i.e., equations of the form

$$B(\tau) = \int_0^\infty K\left(|\tau - \tau'|\right) B(\tau')\, d\tau' + g(\tau). \tag{2}$$

The solution of equation (2) can be represented in the form

$$B(\tau) = g(\tau) + \int_0^\infty \Gamma(\tau', \tau) g(\tau')\, d\tau', \tag{3}$$

where $\Gamma(\tau', \tau)$ is the resolvent corresponding to the kernel $K(\tau)$. To find the resolvent, the relation (for $\tau' > \tau$) was obtained

$$\Gamma(\tau, \tau') = \Phi(\tau' - \tau) + \int_0^\tau \Phi(t)\Phi(t + \tau' - \tau)\, dt, \tag{4}$$

where $\Phi(\tau) = \Gamma(\tau, 0)$. Relation (4) shows that the resolvent of equation (2), $\Gamma(\tau, \tau')$, is expressed through the function $\Phi(\tau)$, depending only on one argument. It follows from (3) and (4) that the solutions of all equations (2), differing in the form of the function $g(\tau)$, are easily represented explicitly through one and the same function $\Phi(\tau)$ corresponding to the given kernel $K(\tau)$.

If the kernel of equation (2) can be represented in the form

$$K(\tau) = \int_a^b A(y) e^{-\tau y}\, dy,$$

then, for finding $\Phi(\tau)$, V. V. Sobolev obtained the relation

$$\int_0^\infty \Phi(\tau) e^{-s\tau}\, d\tau = \frac{\displaystyle\int_a^b A(x) B(0, x)\, \frac{dx}{x+s}}{\displaystyle 1 - \int_a^b A(x) B(0, x)\, \frac{dx}{x+s}}, \tag{6}$$

where $B(0, x)$ is a function determined by the equation

$$B(0, x) = 1 + B(0, x) \int_a^b A(y) B(0, y)\, \frac{dy}{x+y}. \tag{7}$$

The function $\Phi(\tau)$ is found from (6) by inversion of the Laplace transform. V. V. Sobolev also obtained another, linear equation for determining $B(0, x)$, which has the form

$$B(0, x) \left[1 - 2 \int_a^b A(y) \frac{y\, dy}{y^2 - x^2} \right] = 1 - \int_a^b A(y) B(0, y) \frac{dy}{y - x}. \tag{8}$$

Let us note that the function $B(\tau)$ is expressed especially simply in terms of $\Phi(\tau)$ in the case $g(\tau, x) = e^{-\tau x}$, where x is a parameter. Namely, the relation holds

$$B(\tau, x) = B(0, x) e^{-\tau x} \left[1 + \int_0^\tau e^{tx} \Phi(t)\, dt \right]. \tag{9}$$

In the present note, V. V. Sobolev' s method is applied to equation (1). The kernel of this equation can be written in the form

$$K(\tau) = \frac{\alpha}{\pi} \int_1^\infty e^{-\tau y} \frac{dy}{\sqrt{y^2 - 1}}, \tag{10}$$

which corresponds to form (5) with $a = 1$, $b = \infty$, and

$$A(y) = \frac{\alpha}{\pi} \frac{1}{\sqrt{y^2 - 1}}.$$

Using (6), (7), and (8), and also introducing the notation $x = \frac{1}{\eta}$, $y = \frac{1}{\zeta}$, $B(0, x) = \omega(\eta)$, we find

$$\overline{\Phi}(s) = \int_0^\infty \Phi(\tau) e^{-s\tau}\, d\tau = \frac{1}{1 - \dfrac{\alpha}{\sqrt{1 - s^2}}} - \frac{1}{\omega\left(-\dfrac{1}{s}\right)} - 1, \tag{11}$$

where the function $\omega(\eta)$ is determined by the equation

$$\omega(\eta) = 1 + \frac{\alpha}{\pi} \eta \omega(\eta) \int_0^1 \frac{\omega(\zeta)}{\eta + \zeta} \frac{d\zeta}{\sqrt{1 - \zeta^2}}. \tag{12}$$

Let us note that equation (12) determines $\omega(\eta)$ for $0 \le \eta \le 1$. The values of $\omega(\eta)$ for other values of η can be computed using (12).

In inverting (11) we shall assume that τ and α are real. First let $0 < \alpha < 1$. Then it is easy to find that $s = -\sqrt{1 - \alpha^2}$ is a pole and $s = -1$ is a branch point of $\overline{\Phi}(s)$. Taking into account the presence of such singular points and applying the contour-integration method in the inversion, we obtain

$$\Phi(\tau) = \frac{\alpha^2 e^{-\tau\sqrt{1-\alpha^2}}}{\sqrt{1 - \alpha^2}\, \omega\left(\frac{1}{\sqrt{1-\alpha^2}}\right)} + \frac{\alpha}{\pi} \int_1^\infty \frac{e^{-\tau y} \sqrt{y^2 - 1}}{y^2 - (1 - \alpha^2)} \frac{dy}{\omega\left(\frac{1}{y}\right)}. \tag{13}$$

If $-1 < \alpha < 0$, then there is no pole and

$$\Phi(\tau) = \frac{\alpha}{\pi} \int_1^\infty \frac{e^{-\tau y} \sqrt{y^2 - 1}}{y^2 - (1 - \alpha^2)} \frac{dy}{\omega\left(\frac{1}{y}\right)}. \tag{14}$$

Let us consider some special cases. Let, for example, $g(\tau, x) = e^{-\tau x}$. Using (9) and taking into account that $B(0, x) = \omega(\frac{1}{x})$, we have

$$B(\tau, x) = \omega\left(\frac{1}{x}\right) e^{-\tau x} \left[1 + \int_0^\tau e^{tx} \Phi(t)\, dt\right]. \tag{15}$$

Substituting (13) into (15), we find

$$B(\tau, x) = \frac{e^{-\tau x}}{1 - \dfrac{\alpha}{\sqrt{1 - x^2}}} + \omega\left(\frac{1}{x}\right) \left[\frac{\alpha^2}{\sqrt{1 - \alpha^2}\, \omega\left(\dfrac{1}{\sqrt{1 - \alpha^2}}\right)} \frac{e^{-\tau\sqrt{1-\alpha^2}}}{x - \sqrt{1 - \alpha^2}} + \right.$$

$$\left. + \frac{\alpha}{\pi} \int_1^\infty \frac{e^{-\tau y}\sqrt{y^2 - 1}}{y^2 - (1 - \alpha^2)} \frac{dy}{(x - y)\, \omega\left(\frac{1}{y}\right)} \right]. \qquad (16)$$

Substituting (14) into (15), we obtain

$$B(\tau, x) = \frac{e^{-\tau x}}{\sqrt{1 - x^2}} + \omega\left(\frac{1}{x}\right) \frac{\alpha}{\pi} \int_1^\infty \frac{e^{-\tau y}\sqrt{y^2 - 1}}{y^2 - (1 - \alpha^2)} \frac{dy}{(x - y)\, \omega\left(\frac{1}{y}\right)}. \qquad (17)$$

In the special case $x = 1$, relations (16) and (17) give solutions identical to those found earlier for this case by V. A. Fock ([1,2]). Let now

$$g(\tau) = \int_c^d e^{-\tau x} G(x)\, dx. \qquad (18)$$

Then, as follows from equation (1), its solution is represented in the form

$$B(\tau) = \int_c^d G(x) B(\tau, x)\, dx, \qquad (19)$$

where $B(\tau, x)$ is determined by relation (15). This simple case may be of interest, since it corresponds to the run-up on a shore of a wave that is a superposition of plane waves.

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