MATHEMATICS

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1960

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Abstract

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UNIFORM APPROXIMATION OF CONTINUOUS FUNCTIONS ON RIEMANN SURFACES

(Presented by Academician M. A. Lavrent’ ev, 21 X 1959)

S. N. Mergelyan’ s theorem ([1], p. 44) on the approximation of continuous functions by polynomials on closed point sets of the complex plane, which is a generalization of known theorems of M. A. Lavrent’ ev and M. V. Keldysh, carries over to closed Riemann surfaces.

Theorem 1. *In order that a function $f(P)$, defined on a closed set E belonging to a closed Riemann surface R, be expandable in a series of functions rational on R with a single pole at the point Q, converging uniformly to $f(P)$ on E, it is necessary and sufficient that the complement of E consist of one domain containing the point Q, and that $f(P)$ be continuous on E and analytic at every interior point of the set E.*

The necessity of the condition of Theorem 1 is obvious; the sufficiency follows from the following theorem.

Theorem 2. *If the complement of a closed set E relative to a closed Riemann surface R consists of n domains $G_1, \ldots, G_n$, and $Q_1, \ldots, Q_n$ are arbitrary points from these domains, then any single-valued function $f(P)$, continuous on E and analytic at the interior points of E, is expandable in a series of functions rational on R, having poles only at the points $Q_1, \ldots, Q_n$, converging uniformly to $f(P)$ on E.*

For the proof of Theorem 2 one uses the lemma given in the paper [2], which is a transfer of Runge’ s lemma to closed Riemann surfaces:

Lemma 1. *Let D be a domain on a closed Riemann surface R, bounded by a finite number of closed Jordan curves, and let the complement of $\overline{D}$ relative to R consist of n domains $G_1, \ldots, G_n$, and let Q_k be an arbitrary point of G_k $(k = 1, 2, \ldots, n)$. If the function $f(P)$ is analytic on $\overline{D}$, then for any $\varepsilon > 0$ there exists a function $\varphi(P)$, rational on R, with poles only at the points $Q_1, \ldots, Q_n$, satisfying the inequality*

$$\max_{P \in D} |f(P) - \varphi(P)| < \varepsilon.$$

Let $P_1, \dots, P_m$ and Q be arbitrary points of a closed Riemann surface R, $f(P)$ an arbitrary function, and $dg(P)$ an arbitrary differential on R, having poles at the points $P_1, \dots, P_m$; then there exist a function $R(P)$, rational on R, and an abelian differential $dA(P)$ with poles only at the points $P_1, \dots, P_m$ and Q, such that $f_1(P) = f(P) - R(P)$ and $dg_1(P) = dg(P) - dA(P)$ are regular at the points $P_1, \dots, P_m$. It can also be shown that for any closed set E on

on R there exists an Abelian differential of the first kind $d\omega_1(P)$, having no zeros at the boundary points E, and an Abelian differential $d\omega_2(P)$, having no poles at the boundary points E and vanishing on R at the single point Q, which can be chosen arbitrarily.

With the aid of $d\omega_1(P)$ and $R(P)$, Lemma 1 yields an analogous assertion for differentials:

Lemma 2. *Let D be a domain on the closed Riemann surface R, bounded by a finite number of closed Jordan arcs ∂y, and let the complement of $\overline{D}$ with respect to R consist of n domains $G_1, \dots, G_n$, and let Q_k be an arbitrary point of G_k. If the differential $dg(P)$ is analytic on $\overline{D}$, then, for any fixed finite covering of R by cells and for any $\varepsilon > 0$, there exists an Abelian differential $dh(P)$ with poles only at the points Q_k, satisfying the inequality*[1]

$$\max_{P \in D} \left| \frac{dg(P)}{dz} - \frac{dh(P)}{dz} \right| < \varepsilon.$$

Next, almost as in the paper (^{1}), one constructs the averaging of a discontinuous function $f(P)$, defined on the closed set E, and from it, with the aid of $d\omega_1(P)$ and $R(P)$, one obtains an averaging $dH_\delta(P)$ for a differential $dg(P)$ discontinuous on E and analytic at the interior points of E.

Let now D be a domain on the closed Riemann surface R, bounded by a finite number of smooth Jordan arcs; Γ the boundary of D; $dF(P, P_1)$ the elementary differential on R—a differential in P and a function in P_1—with characteristic point Q outside D (see (^{3})); $dF(P, P_1)/dz$ the Cauchy kernel on R. By means of Green' s formula one proves the important formula

$$\varphi(P_1) = \frac{1}{2\pi i} \int_\Gamma \varphi(P)\, dF(P, P_1) - \frac{1}{\pi} \iint_D \frac{dF(P, P_1)}{dz} \frac{\partial \varphi(P)}{\partial \bar{z}}\, dx\, dy, \qquad (1)$$

which is a generalization of the Cauchy formula for nonanalytic, but differentiable, functions, and the analogous formula for covariants

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[1] Here and below $z = x + iy$ denotes the local parameter belonging to the point P. When we have to deal with two variable points, we shall denote the second by P_1, and the corresponding local parameter by $\zeta = \xi + i\eta$.

$$\frac{d\psi(P)}{dz} = \frac{1}{2\pi i}\int_\Gamma \frac{dF(P,P_1)}{dz}\,d\psi(P_1) - \frac{1}{\pi}\iint_D \frac{dF(P,P_1)}{dz}\,\frac{\partial}{\partial\bar\zeta}\left(\frac{d\psi(P_1)}{d\zeta}\right)d\xi\,d\eta. \tag{2}$$

Removing from R an arbitrary closed domain having no common points with E, we obtain an open surface R' containing E. Mapping the universal covering surface $\hat{R}'$ of the surface R' onto a disk, we obtain for R' a fundamental polygon Π and the corresponding Fuchsian group of the second kind. With the aid of automorphic differentials on Π one constructs a differential $dF_{P_1'}(P)$, regular on E and approximating $dF(P,P_1)$ in a neighborhood of the point $P_1 = P_1'$ the better, the closer $P_1' \in E$ is to the boundary of this set.

After this we can prove the following theorem:

Theorem 3. *If the complement of the closed set E with respect to the closed Riemann surface R consists of n domains $G_1, \dots, G_n$ and $Q_1, \dots, Q_n$ are arbitrary points from these domains, while $dg(P)$ is a differential regular at every interior point of E, except for a finite number of points $P_1, \dots, P_m$, where it has poles, and*

continuous on E except at the same points, then, for any fixed finite covering of R by cells and for any $\varepsilon > 0$, there exists an Abelian differential $dh(P)$ on R, with poles only at the points P_i and Q_k, satisfying the inequality

$$\max_{P\in E}\left|\frac{dg(P)}{dz} - \frac{dh(P)}{dz}\right| < \varepsilon.$$

For $m = 0$ the theorem is proved as follows: we average $dg(P)$; to the average $dH_\delta(P)$ and to a domain B with smooth boundary contours, chosen so that the measure* $m(B-E)$ is sufficiently small, we apply formula (2); under the double-integral sign we replace $dF(P,P_1)/dz$ by $dF_{P_1'}(P)/dz$; by Lemma 2 we replace the differentials, regular on $\overline{B}$, thus obtained by Abelian differentials on Q with poles at the points $Q_1, \dots, Q_n$. We obtain an Abelian differential $dh(P)$ satisfying the inequality

$$\max_{P\in E}\left|\frac{dg(P)}{dz} - \frac{dh(P)}{dz}\right| < N_1\omega(\delta) + \frac{N_2}{\delta}m(B-E).$$

For every $\varepsilon > 0$ there exists a δ such that $\omega(\delta) < \varepsilon/2N_1$, and a domain B such that $m(B-E) < \varepsilon\delta/2N_2$.

The general case of Theorem 3 is obtained simply from the case $m = 0$ with the aid of $dA(P)$.

From Theorem 3, with the aid of $d\omega_2(P)$, Theorem 2 is obtained.

Theorem 2, with the aid of the function $R(P)$, is generalized to the case of functions having poles at interior points of E. This theorem is a generalization of Sakakihara' s theorem from the paper (^{2}), where, instead of an arbitrary closed set E, Jordan domains are considered.

With the aid of formula (1) the following theorem is proved.

Theorem 4. *If the planar measure of the continuum E on R is zero, then any function $f(P)$ continuous on E can be represented in the form of a uniformly convergent on E series $\sum_{n=1}^{\infty} R_n(P)$, where $R_n(P)$ are rational functions on R.*

The work was carried out under the supervision of Prof. L. I. Volkovyskii, to whom the author expresses sincere gratitude.

Received
12 X 1959

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[*] The hyperbolic measure associated with the mapping R' onto the unit disk is considered.

Note: Figure translations are in progress. See original paper for figures.

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