ON SEMIGROUPS OF ALMOST IDENTITY TRANSFORMATIONS

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Abstract

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MATHEMATICS

E. G. SHUTOV

ON SEMIGROUPS OF ALMOST IDENTITY TRANSFORMATIONS

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1°. In the present note all stable equivalences are found for the semigroup of all almost identity transformations of an infinite set Ω. Thanks to these results, all endomorphisms of this semigroup distinct from isomorphisms are determined. Further, certain properties of the semigroup of all almost identity partial transformations of the set Ω are investigated. An abstract characterization of this semigroup is given, and all its ideals, normal subsemigroups, and automorphisms are described. In finding the stable equivalences of the semigroup of all almost identity transformations, results of A. I. Mal' tsev are used (^{1}). For finite sets Ω, all the results indicated above have already been obtained ($^{1-3}$).

2°. All notions which are not specially defined in the note are used in their usual sense (see, for example, ($^{1-4}$)).

An equivalence of a semigroup is called **stable** if it is compatible with the multiplication of this semigroup. Let A_1 be a subsemigroup of the semigroup A. We shall say that an equivalence Σ_1 of the semigroup A_1 is a **restriction of an equivalence** Σ of the semigroup A, if elements of the semigroup A are comparable with respect to the equivalence Σ_1 if and only if they are comparable with respect to the equivalence Σ. The identity equivalence of a semigroup and the equivalence under which all elements of the semigroup are comparable with one another are called **trivial equivalences**.

3°. Let Ω be an infinite set. Denote by H_Ω the semigroup of all almost identity transformations of the set Ω, i.e., the totality of all such transformations of the set Ω, each of which moves no more than a finite number of elements, and by S_Ω the group of all one-to-one transformations from H_Ω. If under $a \in H_\Omega$ the set Δ is mapped onto Δ_1, then we shall write $a\Delta = \Delta_1$. The set of all $\alpha \in \Omega$ which the transformation a maps into a single element is called the **contracting complex** of a. Transformations a and b from H_Ω will be called **related** if their contracting complexes are the same and $a\Omega = b\Omega$.

4°. Let $a, b \in H_\Omega$, and let Δ be the set of all those α from Ω, each of which is mapped into itself by the transformations a and b and is the contracting complex of a and b. Denote by $\Pi(a, b)$ the set of all $\alpha \in \Omega$ which are not contained in Δ. Let a and b be related. Then denote by $[a, b] = c$ such a transformation from

S_Ω that for any
$\alpha \in \Pi(a, b)$, $\beta \in a\Pi(a, b)$ one has

$$ca\alpha = b\alpha, \qquad c\beta = \beta.$$

For any $a, b, c \in H_\Omega$ the following properties hold:

1. If a is related to b, and b is related to c, then a is related to c and
$$[b, c][a, b] = [a, c].$$
2. If a, b are related and the defects of ca, a are the same, then cb is related to ca and the transformations $[a, b]$, $[ca, cb]$ are conjugate.
3. If a and b are related and the defects ac, a are the same, then ac is related to bc and $[ac, bc] = [a, b]$.

5^0. **Definition.** Let n be a finite nonnegative integer, and let R be a normal divisor of the group S_Ω. Transformations a and b from H_Ω will be called **comparable modulo R relative to n** if one of the following conditions is satisfied: 1) a and b are equal; 2) the defects of a and b are greater than n; 3) the defects of a and b are equal to n, a and b are related, and $[a, b] \in R$.

By virtue of the properties 4^0, the comparison just defined is a stable equivalence of the semigroup H_Ω. These equivalences, for all possible R and n, will be called **equivalences modulo normal divisors**.

6^0. Let Δ be a finite subset of Ω containing more than four elements. The aggregate P_Δ of all such $a \in H_\Omega$ that $a\Delta \subset \Delta$ and $a\alpha = \alpha$ for every $\alpha \in \Delta$, forms a semigroup isomorphic to the semigroup of all substitutions of the set Δ. Let $\Delta \subset \Delta_1$, where Δ_1 is finite, and let Σ be a stable equivalence of the semigroup P_Δ under which at least two transformations from P_Δ are comparable, the rank of each of which is greater than four. Then, using results of A. I. Mal'cev (^{1}), one can show that the equivalence Σ has the following properties:

1. There exists an equivalence Σ_1 of the semigroup H_Ω modulo a normal divisor (5^0) such that the equivalence Σ is the restriction of the equivalence Σ_1 (2^0).
2. If the equivalence Σ is the restriction of each of the stationary equivalences Σ_1, Σ_2 of the semigroup P_{Δ_1}, then the equivalences Σ_1 and Σ_2 are the same.

7^0. With the aid of properties 1 and 2 (6^0) the following is proved.

Theorem 1. *The only nontrivial stable equivalences of the semigroup H_Ω are the equivalences modulo normal divisors (5^0).*

From the theorem follows

Corollary. The only normal subsemigroups of the semigroup H_Ω are the semigroup H_Ω itself and the normal divisors of the group S_Ω.

This result was obtained by N. N. Vorob' ev (^{4}).

8^0. A homomorphism of a semigroup into itself is called an **endomorphism** of this semigroup. We shall denote by φ_1 the endomorphism of the semigroup H_Ω onto the subsemigroup consisting of one element.

The set P_2, consisting of two such transformations a and b from H_Ω that

$$ab = ba = a^2 = c, \qquad b^2 = b,$$

forms a subsemigroup of H_Ω. The set P_3, consisting of three such transformations a, b, c from H_Ω that

$$ab = ba = ac = ca = a^2 = a, \qquad bc = cb = c, \qquad b^2 = c^2 = b.$$

also forms a subsemigroup of H_Ω. Let S^1 be the set of all even a from S_Ω, and let H^1 be the set of all a from H_Ω of positive defects. Then denote by φ_2 such a mapping of H_Ω onto the subsemigroup P_2 that for any $d_1 \in S_\Omega,\ d_2 \in H^1$ one has

$$\varphi_2 d_1 = b, \qquad \varphi_2 d_2 = a.$$

Next denote by φ_3 such a mapping of H_Ω onto its subsemigroup P_3 that for any $d_1 \in S_\Omega \setminus S^1,\ d_2 \in S^1,\ d_3 \in H^1$ one has

$$\varphi_3 d_1 = b, \qquad \varphi_3 d_2 = c, \qquad \varphi_2 d_3 = a.$$

The mappings φ_2 and φ_3 are endomorphisms of the semigroup H_Ω, and there exist no other endomorphisms of H_Ω onto P_2 and P_3 distinct from φ_2 and φ_3.

Thanks to Theorem 1, the following theorem is proved.

Theorem 2. *The only endomorphisms of the semigroup H_Ω distinct from isomorphisms are endomorphisms of the form $\varphi_1, \varphi_2, \varphi_3$.*

9°. Let Δ_1, Δ_2 be subsets of Ω, which may also be empty. A mapping a of the set Δ_1 into Δ_2 is called a **partial transformation** of the set Ω. We shall denote each of Δ_1, Δ_2, respectively, by $\Pi_1(a)$ and $\Pi_2(a)$. The cardinality of $\Pi_2(a)$ is called the **rank** of a. With respect to the usual multiplication of partial transformations, the totality W_Ω of all partial transformations of the set Ω is a semigroup. The set V_Ω of all $a \in W_\Omega$, each of which moves no more than a finite number α from Ω, forms a subsemigroup of W_Ω, which is called the **semigroup of all almost identical partial transformations**. The totality V_Ω^1 of all a from V_Ω of rank $\leqslant 1$ is the minimal densely embedded ideal of V_Ω.

10°. Let A be a semigroup with zero 0. Denote by A^* the set of all nonzero $a \in A$ such that, for any $b, c \in A$, from $bc = 0$ it follows that $bac = 0$. Let $a \in A$. The set of all $b \in A^*$ for which $ab \neq 0$ will be denoted by A_a.

Theorem 3. *In order that a semigroup A be isomorphic to the semigroup V^1_Ω, it is necessary and sufficient that A have the following properties:*

1. *A contains a zero 0, and the sets Ω, A^* are equipotent.*
2. *For every nonzero $a \in A$ there exists in A^* a unique left identity e_a, and the set A_a is nonempty and finite.*
3. *For every $a \in A^*$ and every nonempty finite subset M of the set A^*, there exists a unique $b \in A$ such that*

$$A_b = M, \qquad e_b = a.$$

4. *If $e_b \in A_a$ $(a, b \in A)$, then $ab \neq 0$.*
5. *If $ab \neq 0$ $(a, b \in A)$, then $A_{ab} = A_b$.*

11°. Thanks to results obtained by L. M. Gluskin (^{3}), the following theorem is valid.

Theorem 4. *In order that a semigroup A be isomorphic to the semigroup V_Ω, it is necessary and sufficient that A have a minimal densely embedded ideal isomorphic to the semigroup V^1_Ω (9°).*

12°. If all elements of the set Δ_1, with the exception, perhaps, of a finite number of them, are contained in the set Δ_2, then we shall write $\Delta_1 \sigma \Delta_2$.

13°. Let $a \in V_\Omega$, let Δ be a subset of Ω, and let $\Delta_1 = \Delta \cap \Delta_2$, where Δ_2 is the set of all $\alpha \in \Pi_1(a)$ such that $a\alpha = \alpha$, and if $a\beta = \alpha$, then $\beta = \alpha$. We shall denote each of the sets $\Delta \setminus \Delta_1$ and $\Pi_2(a) \setminus \Delta_1$, respectively, by $\Gamma(\Delta, a)$ and $\Gamma(a, \Delta)$.

Definition. Let Δ be a subset of Ω, and let n be a finite nonnegative integer not exceeding the cardinality of the set $\Omega \setminus \Delta$. By the set $V(\Delta, n)$ we shall mean the set of all $a \in V_\Omega$ such that:

1. $\Pi_1(a)\sigma\Delta$ (12°),
2. If the sets $\Gamma(\Delta, a)$ and $\Gamma(a, \Delta)$ have cardinalities p and q, respectively, then $q \leqslant p + n$.

Obviously, if Δ contains a finite number m of elements, then $V(\Delta, n)$ is the totality of all such a from V_Ω whose ranks do not exceed $n + m$.

Each set $V(\Delta, n)$, for arbitrary Δ and n, is an ideal of the semigroup V_Ω.

Theorem 5. *All ideals of the semigroup V_Ω are exhausted by set-theoretic sums of ideals of the form $V(\Delta, n)$.*

14° Definition. Let Ω^* be any set whose elements are distinct subsets Δ_i of the set Ω, and let $\Omega_a = \Omega \setminus \Pi_1(a)$, where $a \in V_\Omega$. By a set $V(\Omega^*)$ we shall mean the collection of all such a from V_Ω for each of which there exists in Ω^* a finite number of such Δ_i, whose union is Δ, such that $\Omega_a \subset \Delta$ (12^0).

In order to exhaust all possible sets $V(\Omega^*)$, it is enough to assume that Ω^* either consists of one Δ_1 (finite or infinite), or of an infinite number of such infinite Δ_i that the union of any finite number of them differs from the union of all sets from Ω^* by an infinite number of elements.

Every normal divisor of the group S_Ω (3^0) and every set of the form $V(\Omega^*)$ are normal subsemigroups of the semigroup V_Ω.

Theorem 6. *All normal subsemigroups of the semigroup* V_Ω *are exhausted by the normal divisors of the group* S_Ω *and by all possible subsemigroups of the form* $V(\Omega^*)$.

15^0. **Theorem 7.** *Every automorphism of the semigroup* V_Ω *is induced by a unique automorphism of the semigroup* W_Ω (9^0).

The only automorphisms of the semigroup W_Ω are inner automorphisms ([3]); therefore Theorem 7 gives a complete description of all automorphisms of the semigroup V_Ω.

Since not all invertible elements of the semigroup W_Ω belong to V_Ω, it follows, according to Theorem 7, that the semigroup V_Ω has automorphisms which are not inner.

Udmurt State
Pedagogical Institute
named after the Decade of the UASSR

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