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**Abstract**

**Full Text**

## Reports of the Academy of Sciences of the USSR

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**MATHEMATICS**

**V. G. Lemlein**

### ON THE GEOMETRIC MEANING OF THE PROJECTIVE CURVATURE TENSOR IN MANIFOLDS WITH AFFINE CONNECTION

*(Presented by Academician P. S. Aleksandrov, 13 IV 1960)*

1. Let an object of affine connection ( $\Gamma_{jk}^p = \Gamma_{kj}^p$ ) be given on a differentiable manifold  $\{V^n\}$ ; then in each local centro-projective space  $\{P^n\}$  <sup>(4)</sup> there arises an invariant hyperquadric <sup>(5)</sup>

$$\varphi \equiv \left[ \frac{1}{(n+1)^2} \Gamma_{ai}^a(x) \Gamma_{bj}^b(x) - \frac{\delta_{ij}(x)}{(n-1)} \right] u^i u^j - \frac{2}{(n+1)} \Gamma_{ai}^a(x) u^i + 1 = 0, \quad (1)$$

where  $\delta_{ij}(x)$  is the symmetric part of the Ricci tensor.

Each  $\{P^n\}$  may be regarded as a symmetric projective-Euclidean space <sup>(3)</sup>, and under the additional condition  $\det \|\delta_{ij}(x)\| \neq 0$ —as a space of constant curvature.

2. The connection  $\Gamma_{jk}^p(x, u)$  arising in each  $\{P^n\}$  can be naturally defined by the polar correlative correspondence with respect to the hyperquadric (1) <sup>(2,3)</sup>.

Indeed, taking  $(\tilde{u}^1, \tilde{u}^2, \dots, \tilde{u}^n, \tilde{u}^0)$  as homogeneous coordinates of a point in  $\{P^n\}$  and considering them as functions of the parameters  $(t^1, t^2, \dots, t^n)$ , we can associate with each point of  $\{P^n\}$   $n$  points

$$v_i^\alpha = \frac{\partial \tilde{u}^\alpha}{\partial t^i} + \mu_i \tilde{u}^\alpha \quad (\alpha = 0, 1, 2, \dots, n; i = 1, 2, \dots, n).$$

If we now require that these points lie in the polar hyperplane of the point  $(\tilde{u}^\alpha)$  with respect to the hyperquadric (1), then we obtain

$$v_i^\alpha = \frac{\partial \tilde{u}^\alpha}{\partial t^i} - \frac{\partial \ln \sqrt{\tilde{u}^{02} \varphi}}{\partial t^i} \tilde{u}^\alpha.$$

Normalizing the homogeneous coordinates so that  $\tilde{u}^0 = 1$ , and taking  $(u^1, u^2, \dots, u^n)$  as parameters, we obtain

$$\begin{aligned} v_i^j &= \delta_i^j - \frac{\partial \ln \sqrt{\varphi}}{\partial u^i} u^j, \\ v_i^0 &= -\frac{\partial \ln \sqrt{\varphi}}{\partial u^i}. \end{aligned} \quad (2)$$

Further, to each point  $(v_i^\alpha)$ , in turn, there will correspond  $n$  points

$$w_{kl}^\beta = \frac{\partial v_k^\beta}{\partial u^l} - \frac{\partial \ln \sqrt{\varphi}}{\partial u^l} v_k^\beta.$$

Representing them in the form  $w_{kl}^\beta = \Gamma_{kl}^j v_j^\beta + \Gamma_{kl}^0 u^\beta$  and taking account of relations (2), we shall have

$$\begin{aligned} &\Gamma_{kl}^j \left( \delta_j^i - \frac{\partial \ln \sqrt{\varphi}}{\partial u^j} u^i \right) + \Gamma_{kl}^0 u^i = \\ &= -\frac{\partial^2 \ln \sqrt{\varphi}}{\partial u^k \partial u^l} u^i + \frac{\partial \ln \sqrt{\varphi}}{\partial u^k} \frac{\partial \ln \sqrt{\varphi}}{\partial u^l} u^i - \delta_k^i \frac{\partial \ln \sqrt{\varphi}}{\partial u^l} - \delta_l^i \frac{\partial \ln \sqrt{\varphi}}{\partial u^k}, \\ &-\Gamma_{kl}^j \frac{\partial \ln \sqrt{\varphi}}{\partial u^j} + \Gamma_{kl}^0 = -\frac{\partial^2 \ln \sqrt{\varphi}}{\partial u^k \partial u^l} + \frac{\partial \ln \sqrt{\varphi}}{\partial u^k} \frac{\partial \ln \sqrt{\varphi}}{\partial u^l}. \end{aligned}$$

Hence we find the connection

$$\Gamma_{kl}^i(x, u) = -\delta_k^i \frac{\partial \ln \sqrt{\varphi}}{\partial u^l} - \delta_l^i \frac{\partial \ln \sqrt{\varphi}}{\partial u^k} \quad (3)$$

of the symmetric projective-Euclidean space.

3. Substituting into (3) the value of  $\varphi$  from (1), we obtain

$$\begin{aligned} \Gamma_{kl}^i(x, u) &= -\left\{ \delta_k^i \left[ \left( \frac{1}{(n+1)^2} \Gamma_{al}^a(x) \Gamma_{bp}^b(x) - \frac{\sigma_{lp}(x)}{(n-1)} \right) u^p - \frac{1}{(n+1)} \Gamma_{al}^a(x) \right] \right. \\ &+ \delta_l^i \left[ \left( \frac{1}{(n+1)^2} \Gamma_{ak}^a(x) \Gamma_{bq}^b(x) - \frac{\sigma_{kq}(x)}{(n-1)} \right) u^q - \frac{1}{(n+1)} \Gamma_{ak}^a(x) \right] \left. \right\} : \\ &: \left\{ \left[ \frac{1}{(n+1)^2} \Gamma_{ap}^a(x) \Gamma_{bq}^b(x) - \frac{\sigma_{pq}(x)}{(n-1)} \right] u^p u^q - \frac{2}{(n+1)} \Gamma_{ap}^a u^p + 1 \right\}. \end{aligned}$$

Hence, for  $u^i = 0$ , we have

$$\Gamma_{kl}^i(x, 0) = \frac{1}{(n+1)} \delta_k^i \Gamma_{al}^a(x) + \frac{1}{(n+1)} \delta_l^i \Gamma_{ak}^a(x),$$

$$\Pi_{al}^a(x, 0) = \Pi_{al}^a(x).$$

The curvature tensor and the Ricci tensor in  $\{P^n\}$  take, respectively, the form

$$r_{k,ji}^h(x, u) = \frac{1}{(1-n)} [\delta_j^h r_{ik}(x, u) - \delta_i^h r_{jk}(x, u)],$$

$$r_{ik}(x, u) = (1-n) \frac{1}{\sqrt{\varphi}} \frac{\partial^2 \sqrt{\varphi}}{\partial u^i \partial u^k},$$

where, for  $u^i = 0$ , we have  $r_{ik}(x, 0) = \sigma_{ik}(x)$  and, consequently,

$$r_{k,ji}^h(x, 0) = \frac{1}{(1-n)} \delta_j^h \sigma_{ik}(x) - \frac{1}{(1-n)} \delta_i^h \sigma_{jk}(x). \quad (4)$$

4. Let us now consider the projective curvature tensor of the original affine-connection space

$$\bar{P}_{k,ji}^h(x) = R_{k,ji}^h(x) - 2P_{[ij]}(x) \delta_k^h + 2\delta_{[i}^h P_{j]k}(x),$$

where

$$P_{ij}(x) = \frac{1}{(1-n^2)} [(n+1)R_{ij} + V_{ij}].$$

The vanishing of this tensor for  $n > 2$  gives the necessary and sufficient condition that the space be projective-Euclidean <sup>(1)</sup>.

Expressing the Ricci tensor through its symmetric part and the tensor of equiaffinity

$$R_{ij}(x) = \frac{V_{ji}(x)}{2} + \sigma_{ij}(x),$$

we shall have

$$P_{ij}(x) = -\frac{1}{2(n+1)} V_{ij}(x) + \frac{1}{(1-n)} \sigma_{ij}(x)$$

and, consequently,

$$\begin{aligned}
 P^h_{k,ji}(x) &= R^h_{k,ji}(x) - \frac{1}{(1-n)} (\delta_i^h \sigma_{jk}(x) - \delta_j^h \sigma_{ik}(x)) + \\
 &+ \frac{3}{2(n+1)} [\delta_{(j}^h V_{i)k}(x) - V_{ji}(x) \delta_k^h].
 \end{aligned} \tag{5}$$

Introducing the tensor

$$Q^h_{k,ji}(x) = \frac{3}{2(n+1)} [\delta_{(j}^h V_{i)k}(x) - V_{ji}(x) \delta_k^h] \tag{6}$$

and taking (4) into account, relation (5) can be rewritten in the form

$$P^h_{k,ji}(x) - Q^h_{k,ji}(x) = R^h_{k,ji}(x) - r^h_{k,ji}(x, 0). \tag{7}$$

Hence we obtain the theorem:

**Theorem.** *The difference between the projective-curvature tensor and the tensor (6), which depends only on the equiaffinity tensor, at each point of the manifold under consideration is equal to the difference between the curvature tensor of the original affine connection and the curvature tensor of the local space computed at the corresponding point.*

5. The tensor (6) vanishes if and only if the space is equiaffine. In this case (7) takes the form

$$P^h_{k,ji}(x) = R^h_{k,ji}(x) - r^h_{k,ji}(x, 0).$$

Further, for projective-Euclidean spaces we have

$$-Q^h_{k,ji}(x) = R^h_{k,ji}(x) - r^h_{k,ji}(x; 0), \tag{8}$$

and for equiaffine projective-Euclidean spaces

$$R^h_{k,ji}(x) = r^h_{k,ji}(x, 0). \tag{9}$$

If, in the last case, one further assumes the nondegeneracy of the Ricci tensor of the original affine connection, then it becomes possible to introduce the scalar

$$K(x) = \frac{\varepsilon}{(n-1)} \sqrt[n]{\frac{\det \|R_{ij}(x)\|}{a^2(x)}},$$

where  $\varepsilon$  is an  $n$ -th root of unity, and  $a(x)$  is a relative scalar of unit weight, determined up to a constant factor from the condition

$$\Gamma^a_{ai}(x) = \frac{\partial \ln a(x)}{\partial x^i}.$$

The local spaces now become spaces of constant curvature  $K(x)$  with metric tensors

$$g_{ij}(x, u) = \varepsilon r_{ij}(x, u) \sqrt[n]{\frac{a^2(x)}{\det \|R_{pq}(x)\|}}.$$

If in an equiaffine projective-Euclidean space with nondegenerate Ricci tensor one takes as the metric tensor the

tensor

$$g_{ij}(x, 0) = \varepsilon R_{ij}(x) \sqrt[n]{\frac{a^2(x)}{\det \|R_{pq}(x)\|}},$$

which, generally speaking, is not covariantly constant in the affine connection under consideration; then the scalar  $K(x)$  can be defined in the same way as the curvature in a two-dimensional direction.

It is expedient to study the distinguished type of spaces of affine connection, which are closest to spaces of constant curvature, in special coordinates (6).

For two-dimensional spaces the tensor of projective curvature (5) vanishes identically, and, consequently, relation (7) takes the form (8), and in the case of an equiaffine connection, (9).

6. The tensor of projective curvature (5) and the tensor (6) may be expressed through the components of the connecting object (7) and their first derivatives by the formulas

$$\begin{aligned} P^h_{k,ji}(x) &= G^h_{k,ji}(x) - \frac{1}{(1-n)} \left[ \delta^h_j G^{\alpha}_{(i,k)\alpha}(x) - \delta^h_i G^{\alpha}_{(j,k)\alpha}(x) \right] + \\ &+ \frac{3}{2(n+1)} \left[ G^{\alpha}_{\alpha,(ji)}(x) \delta^h_k - G^{\alpha}_{\alpha,ji}(x) \delta^h_k \right], \end{aligned} \quad (10)$$

$$Q^h_{k,ji}(x) = \frac{3}{2(n+1)} \left[ G^{\alpha}_{\alpha,(ji)}(x) \delta^h_k - G^{\alpha}_{\alpha,ji}(x) \delta^h_k \right], \quad (11)$$

despite the fact that the quantities

$$G^h_{k,ji}(x) = \frac{\partial G^h_{ik}(x)}{\partial x^j} - \frac{\partial G^h_{jk}(x)}{\partial x^i} + G^{\alpha}_{ik}(x) G^h_{j\alpha}(x) - G^{\alpha}_{jk}(x) G^h_{\alpha i}(x).$$

do not form a tensor.

Rewriting now relation (7) in the form

$$G_{k,ji}^h(x) - \frac{1}{(1-n)} \left[ \delta_j^h G_{(i,k)\alpha}^\alpha(x) - \delta_i^h G_{(j,k)\alpha}^\alpha(x) \right] = R_{k,ji}^h(x) - r_{k,ji}^h(x, 0),$$

we note that the difference between the curvature tensor of the affine connection under consideration and the curvature tensor of the local space, calculated at the corresponding point, is also expressed through the components of the connecting object and their first derivatives; moreover, the tensors (10) and (11), and hence also their difference, do not depend on the choice of the fundamental relative scalar  $a(x)$  of weight  $N = 1$ , which determines the decomposition of the affine connection (7):

$$\Gamma_{jk}^i(x) = \gamma_{jk}^i(x) + G_{jk}^i(x),$$

$$\gamma_{jk}^i(x) = \frac{1}{(n+1)} \delta_j^i \frac{\partial \ln a}{\partial x^k} + \frac{1}{(n+1)} \delta_k^i \frac{\partial \ln a}{\partial x^j}.$$

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*Note: Figure translations are in progress. See original paper for figures.*

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