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Abstract

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MATHEMATICS

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ON THE DISTRIBUTION OF PRIME k-TWINS

(Presented by Academician I. M. Vinogradov on 25 II 1960)

§ 1. The laws of distribution of pairs $p,\ p+2k$ of prime k-twins, for a given integer $k \geq 1$, are very difficult and at the present time have been little studied. By the method of trigonometric sums of I. M. Vinogradov (^{1}), for the number of pairs of k-twins belonging to an interval of given length, it is possible to prove asymptotic relations for almost all integers $k \geq 1$. In the present communication new theorems on k-twins are given, obtained by combining the aforementioned result of I. M. Vinogradov' s method with results of A. Selberg' s sieve method, namely, theorems concerning the difference between consecutive pairs of prime k-twins.

Theorem 1. The assertion that the inequality

$$\left|p_{ki} - p_{k(i-1)}\right| > \frac{\ln^2 p_{ki}}{f(p_{ki})}$$

holds for almost all neighboring pairs $[p_{ki}; p_{ki} + 2k]$, $[p_{k(i-1)}; p_{k(i-1)} + 2k]$ of prime k-twins, where $f(t) \to \infty$ is any function such that $\ln^2 t/f(t)$ increases monotonically, is true for almost all integers $k \geq 1$.

A somewhat more precise result, explicitly connected with the length of the interval, is supplied by Theorem 2.

Theorem 2. For almost all neighboring pairs $[p_{ki}; p_{ki}+2k]$, $[p_{k(i-1)}; p_{k(i-1)}+2k]$ of prime k-twins from the interval $(0, x)$, for each $2 \leq 2k \leq x/\ln x$, excluding no more than $cx/(\ln x)^M$ of them, where $M > 1$ is arbitrary fixed and c depends only on M, the inequality

$$\left|p_{ki} - p_{k(i-1)}\right| > \frac{\ln^2 p_{ki}}{\alpha_k\, \omega(p_{ki})}, \qquad \alpha_k = \prod_{\substack{p|k \\ p>5}} \frac{p-2}{p-4},$$

is valid, where $\omega(t) \to \infty$ is any function such that $\ln^2 t/\omega(t)$ increases monotonically, and p runs through the prime divisors of k.

The following assertion, which is an analogue of a theorem of P. Erdős (^{2}), gives a characterization of irregularity in the distribution of pairs of prime numbers $p,\ p+2k$.

Theorem 3. Let $x > x_0$; $E(x) = \ln^2 x/\omega(x)$; $0 < D(x) \leq E(x)$, where $\omega(x) \to \infty$ as $x \to \infty$. Then for almost every integer $2 \leq 2k \leq x/\ln x$ there exist

$$N = \left[\alpha_k \frac{\ln^2 x}{D(x)}\right], \qquad \alpha_k = \beta \prod_{\substack{p|k \\ p>3}} \frac{p-4}{p-2},$$

consecutive pairs

$$[p_{ki}; p_{ki} + 2k], \dots, [p_{k(i+N)}; p_{k(i+N)} + 2k]$$

of prime k-twins, lying in the interval $(0, x)$, such that

$$p_{k(i+s)} - p_{k(i+s-1)} > D(x) \qquad (s = 1, 2, \dots, N),$$

where $\beta > 0$ is an absolute constant.

The obtained order of growth of the quantity N is apparently close to the final one.

§ 2. The derivation of the formulated propositions is based on the following theorem A, obtained by the author by the method of I. M. Vinogradov.

Theorem A. *For every integer $2 \leq 2k \leq x/\ln x$, with the exception of no more than $cx/(\ln x)^M$ of them, where $M > 1$ is arbitrary fixed and c depends only on M, the number $\pi_k(x)$ of pairs of prime k-twins from the interval $(0, x)$ is expressed in the form*

$$\pi_k(x) = 2 \prod_{p>2} \frac{p(p-2)}{(p-1)^2} \frac{x}{\ln^2 x} \prod_{\substack{p|k \\ p>2}} \frac{p-1}{p-2} + O\left(\frac{x}{\ln^3 x}\right).$$

Let us indicate the main points of the proof of the theorems.

Let x be sufficiently large; $F(x) = \ln^2 x/\omega(x)$. Choose an arbitrary number $2k$ under the condition $F(x) < 2k \leq x/\ln x$ and not exceptional in the sense of theorem A.

The set $\mathfrak{M}$ of all pairs

$$[p_{k1},\ p_{k1} + 2k], \dots, [p_{ks};\ p_{ks} + 2k]$$

of prime k-twins lying on the segment $(0, x)$, we split into two subsets $\mathfrak{M}_1$ and $\mathfrak{M}_2$:

$$[p_{ki};\, p_{ki} + 2k] \in \mathfrak{M}_1, \quad \text{if } p_{ki} - p_{k(i-1)} > F(x);$$

the complement of $\mathfrak{M}_1$ in $\mathfrak{M}$ we denote by $\mathfrak{M}_2$. We estimate from above the number of numbers from $\mathfrak{M}_2$.

We have

$$\sum_{\mathfrak{M}_2} 1 = \sum_{0<2n\le F(x)} \sum_{\substack{2n = p_{ki} - p_{k(i-1)} \\ 2\le i\le s}} 1.$$

Next we note that the inner sum is not smaller than the number of solutions of the system of equations

$$p_2 = p_1 - 2k; \qquad p_3 = p_1 - 2n; \qquad p_4 = p_1 - 2(n+k)$$

in primes p_i $(i = 1, \dots, 4)$, belonging to the interval $(0, x)$, so that, applying the result of the sieve of A. Selberg $(^3)$, we shall have

$$\sum_{\mathfrak{M}_2} 1 \ll \frac{x}{\ln^4 x} \sum_{2<2n\le F(x)} \prod_p \frac{p - \gamma(p)}{p\left(1 - \frac{1}{p}\right)^4}, \tag{1}$$

where $\gamma(p)$ is the number of residue classes to which the numbers $0, 2k, 2n, 2(n+k)$ belong modulo the prime modulus p. Putting now $\Delta = kn(k^2 - n^2)$, we find that for $p \mid \Delta$, $1 \le \gamma(p) \le 3$, and for $p \nmid \Delta$, $\gamma(p) = 4$. Therefore for the product from (1) we have

$$\prod_p \ll \prod_{\substack{p\mid k \\ p>2}} \frac{p-1}{p-2} \prod_{\substack{p\mid k \\ p>3}} \frac{p-2}{p-4} \prod_{\substack{p\mid n(k^2-n^2) \\ p\nmid k,\ p>3}} \frac{p-1}{p-4}. \tag{2}$$

Introducing the notation:

$$\Pi_a = \prod_{p\mid a} \left(1 + \frac{15}{p}\right)^4; \qquad Q(x) = \sum_{2k<t\le 2k+F(x)} \Pi_t; \qquad V(x) = \sum_{2k-F(x)\le t\le 2k} \Pi_t$$

and T_k instead of the first two products in (2), we obtain for the sum on the right in relation (1) the estimate

$$\ll T_k\,(Q(x)V(x))^{1/4} \left[\sum_{2\le n\le F(x)} (\Pi_n)^{1/2}\right]^{1/2} \ll T_k\,(F(x)F(x))^{1/4}(F(x))^{1/2}. \tag{3}$$

Combining the estimates (1)–(3), we find

$$\sum_{\mathfrak{M}_2} 1 \ll \frac{x}{\ln^2 x \cdot \omega(x)} \prod_{\substack{p/k \\ p>3}} \frac{p-1}{p-4}.$$

Taking now into account the relation

$$\sum_{\mathfrak{M}_1} 1 + \sum_{\mathfrak{M}_2} 1 = \pi_k(x),$$

by virtue of Theorem A we obtain Theorem 2, and from this latter, owing to the fact that $\prod \frac{p-2}{p-4}$ is on average a constant, Theorem 1 follows.

§ 3. For the derivation of Theorem 3 the same means suffice. The idea of the derivation is as follows. Instead of the function $F(x)$, introduced in § 2, we take the function $D(x)$

$$0 < D(x) \ll \frac{\ln^2 x}{\omega(x)}.$$

In this case it turns out that the set $\mathfrak{M}_1$ is nonempty, while for the number $T(x; D(x))$ of numbers of the set $\mathfrak{M}_2$ the estimate

$$T(x; D(x)) \ll \frac{xD(x)}{\ln^4 x} \prod_{\substack{p/k \\ p>3}} \frac{p-1}{p-4}$$

is valid.

Further, to each sequence of prime numbers $p_{k(s+1)}, \cdots, p_{k(s+r)}$ belonging to $\mathfrak{M}_1$ there corresponds a number $p_{ks} \in \mathfrak{M}_2$. Denoting by N the maximal length of such a sequence from $\mathfrak{M}_1$, we shall have

$$\pi_k(x) < 2NT(x; D(x)).$$

Hence, applying Theorem A, we obtain

$$N > \beta \frac{\ln^2 x}{\alpha_{kD}(x)}, \qquad \alpha_k = \prod_{\substack{p/k \\ p>3}} \frac{p-2}{p-4},$$

where $\beta > 0$ is an absolute constant, which proves the theorem.

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CITED LITERATURE

[1] I. M. Vinogradov, *Selected Works*, Publishing House of the Academy of Sciences of the USSR, 1952. [2] P. Erdős, *Acta math. Szeged*, 13, 57 (1949). [3] N. I. Klimov, *UMN*, 13, no. 3 (81), 146 (1958).

Note: Figure translations are in progress. See original paper for figures.

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