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Abstract

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MATHEMATICS

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BOUNDARY PROPERTIES OF FUNCTIONS FROM “WEIGHTED” CLASSES

(Presented by Academician S. L. Sobolev, 20 I 1960)

In the dissertation of L. D. Kudryavtsev ⁽¹⁾, relations were considered between the functional classes $H_p^{(r)}$, the theory of which was developed by S. M. Nikol'skii ^(2, 3), and the weighted spaces $W_{p,\alpha}^{(l)}$. In our preceding work ⁽⁴⁾, characteristic boundary properties of functions from $W_{p,\alpha}^{(l)}$ were studied. For $p = 2$, a similar investigation had earlier been carried out in the work of A. A. Vasharin ⁽⁵⁾. In the present note we set forth the results of further investigations in this direction. These results are formulated in terms of the generalized spaces of S. L. Sobolev $W_p^{(r)}$ ($r \geq 0$, not necessarily an integer ⁽⁶⁾), and therefore, alongside the consideration of boundary values of functions from weighted classes, questions naturally arise that are connected with the theory of the spaces $W_p^{(r)}$. The connection between the spaces W and weighted classes was discovered in the work of V. I. Kondrashov ⁽⁷⁾. The fruitfulness of such a connection in the study of the spaces $W_p^{(r)}$ is beyond doubt—we point to the work of S. V. Uspenskii ⁽⁸⁾, where, by means of this connection, embedding theorems analogous to the theorems of S. M. Nikol'skii ⁽²⁾ in the classes $H_p^{(r)}$ were obtained.

Let E_n be Euclidean space of n dimensions, $X(x_1, \dots, x_n)$ a point in it; E_{n-1} the subspace $x_n = 0$, $x(x_1, \dots, x_{n-1})$ a point in it;

$$E_n^\delta \text{ is the half-space } x_n > \delta; \quad |X - Y| = \left[\sum_1^n (x_i - y_i)^2 \right]^{1/2};$$

l is a positive integer; $W_p^{(l)}$ is the space of S. L. Sobolev ⁽⁹⁾. We shall say that the function $U(X)$ belongs to the weighted class $W_{p,\alpha}^{(l)}(E_n^0)$ if:

- a) $U(X) \in W_p^{(l)}(E_n^\delta)$ for every $\delta > 0$;
- b)

$$D_{p,\alpha}^l(U) = \int_{E_n^0} x_n^\alpha \sum_{\alpha_1 + \dots + \alpha_n = l} \left| \frac{\partial^l U(X)}{\partial x_1^{\alpha_1} \dots \partial x_n^{\alpha_n}} \right|^p dX < \infty.$$

By definition, the function $U(X)$ belongs to $W_p^{(r)}(E_n^0)$ ($r \geq 0$, $1 \leq p < \infty$) if it belongs to $W_p^{[r]}(E_n^0)$ ($[r]$ is the integer part of r), and, when r is noninteger, in addition its highest derivatives satisfy the relation

$$L_p^r(\partial^{[r]}U) = \int_{E_n^0} dY \int_{E_n^0} \frac{|\partial^{[r]}U(X) - \partial^{[r]}U(Y)|^p}{|X - Y|^{n+p(r-[r])}} dX < \infty.$$

Here

$$\partial^{[r]}U = \partial_{l_1 \dots l_n}^{[r]} U = \frac{\partial^{[r]}U}{\partial x_1^{l_1} \dots \partial x_n^{l_n}}$$

is any of the partial derivatives of order $[r]$. The definition for the whole space (and also for a domain) is analogous ⁽⁶⁾. The norm in the space $W_p^{(r)}(E_n^0)$ is given, for integer r , by

as usual ⁽⁹⁾, and for noninteger r as follows:

$$\|U\|_{W_p^{(r)}(E_n^0)} = \left\{ \|U\|_{W_p^{[r]}(E_n^0)}^p + \sum_{l_1 + \dots + l_n = [r]} L_p^r(\partial_{l_1 \dots l_n}^{[r]} U) \right\}^{1/p}.$$

Finally, let E_{n-m} be the subspace $x_{n-m+1} = \dots = x_n = 0$, let $x'(x_1, \dots, x_{n-m})$ be a point in it, and let $x''(x_{n-m+1}, \dots, x_n) \in E_m$ be a point of the orthogonal complement ($E_{n-m} \oplus E_m = E_n$). Let $C^\delta(E_{n-m})$ be the “cylinder” of radius δ , whose “axis” is the subspace E_{n-m} , i.e., the set of points $X \in E_n$ for which

$$|x''| = \sqrt{x_{n-m+1}^2 + \dots + x_n^2} \leq \delta.$$

By definition, the function $U(X)$ belongs to the weighted class $W_{p,\alpha}^{(l)}(E_n - E_{n-m})$ if:

- a) $U(X) \in W_p^{(l)}[E_n - C^\delta(E_{n-m})]$ for every $\delta > 0$;
- b)

$$D_{p,\alpha}^l(U, E_n - E_{n-m}) = \int_{E_n} |x''|^\alpha \sum_{l_1 + \dots + l_n = l} \left| \frac{\partial^l U(X)}{\partial x_1^{l_1} \dots \partial x_n^{l_n}} \right|^p dX < \infty.$$

In the case $m = 1$, obviously, $x \equiv x'$, $x'' \equiv x_n$; however, as will be seen below, one has to distinguish the classes $W_{p,\alpha}^{(l)}(E_n^0)$ and $W_{p,\alpha}^{(l)}(E_n - E_{n-1})$.

Theorem 1. *Let*

$$U(X) \in \begin{cases} W_{p,\alpha}^{(l)}(E_n^0), & \text{when } m = 1; \\ W_{p,\alpha}^{(l)}(E_n - E_{n-m}), & \text{when } 2 \leq m < n, \end{cases}$$

and let α be representable in the form $\alpha = \alpha_0 + pk$, where $0 < \alpha_0 + m < p$; k is one of the numbers $0, 1, \dots, (l-1)$ (i.e., the number α satisfies the inequality $-m < \alpha < pl - m$ and does not assume the values $p - m, 2p - m, \dots$). Then, in the sense of convergence almost everywhere in x' and simultaneously in the sense of convergence in the metric L_p , there exist limits

$$\lim_{x'' \rightarrow 0} \frac{\partial^i U(X)}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} = \varphi_{i_1, \dots, i_n}^i(x'), \quad i = 0, 1, \dots, l - k - 1; \quad 0 \leq i_1 + \dots + i_n = i.$$

Moreover, the following assertions are valid:

a)

$$\varphi_{0, \dots, 0, i_{n-m+1}, \dots, i_n}^i \in W_p^{(r_i)}(E_{n-m}), \quad r_i = l - k - i - \frac{\alpha_0 + m}{p};$$

b)

$$\left\| \varphi_{0, \dots, 0, i_{n-m+1}, \dots, i_n}^i \right\|_{W_p^{(r_i)}(E_{n-m})}^p \leq \begin{cases} c \left\{ \|U\|_{W_p^{(l-k-1)}(E_n^0)}^p + D_{p,\alpha}^l(U, E_n^0) \right\}, & m = 1; \\ c \left\{ \|U\|_{W_p^{(l-k-1)}(E_n)}^p + D_{p,\alpha}^l(U, E_n - E_{n-m}) \right\}, & 2 \leq m < n; \end{cases}$$

$$0 \leq i_{n-m+1} + \dots + i_n = i \leq l - k - 1;$$

c)

$$\frac{\partial^j \varphi_{0, \dots, 0, i_{n-m+1}, \dots, i_n}^i}{\partial x_1^{j_1} \dots \partial x_{n-m}^{j_{n-m}}} = \varphi_{j_1, \dots, j_{n-m}, i_{n-m+1}, \dots, i_n}^{i+j}, \quad 0 \leq j_1 + \dots + j_{n-m} = j \leq l - k - i - 1.$$

Note that, for functions of the class $W_{p,\alpha}^{(l)}(E_n - E_{n-1})$, the limiting values on E_{n-1} as $x_n \rightarrow +0$ and $x_n \rightarrow -0$ are, generally speaking, different; for this reason it is necessary, in the formulation given above, to single out the case $m = 1$ (see also Theorem 5). When the weight disappears ($\alpha = 0$), the theorem formulated constitutes part of the well-known embedding theorem of S. L. Sobolev–V. I. Kondrashov ^(9,11,12).

We note that assertions similar to those stated in Theorem 1, concerning the existence of boundary values and their properties, but in terms of the classes $H_p^{(r)}$ on the boundary, were first proved by L. D. Kudryavtsev ⁽¹⁾. However

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The existence of limiting values on E_{n-1} in the mean for functions from weighted classes for $p = 2$ was considered in ⁽¹⁰⁾.

namely, in terms of the classes $W_p^{(r)}$ it proves possible to characterize completely the boundary values of functions from weighted classes, which is the inverse of Theorem 1.

Theorem 2. Let on the subspace E_{n-m} , $1 \leq m < n$, there be given a system of functions

$$\varphi_{i_{n-m+1}, \dots, i_n}^i(x') \in W_p^{(l-i-1+r)}(E_{n-m}), \quad 0 < r < 1,$$

$$0 \leq i_{n-m+1} + \dots + i_n = i \leq l - 1.$$

In the space E_n there exists a function $U(X)$ possessing the properties:

a) $U(X) \in W_{p,\alpha}^l(E_n - E_{n-m})$, $\alpha = p(1 - r) - m$;

b)

$$\left. \frac{\partial^i U(X)}{\partial x_{n-m+1}^{i_{n-m+1}} \dots \partial x_n^{i_n}} \right|_{x''=0} = \varphi_{i_{n-m+1}, \dots, i_n}^i(x');$$

c) $U(X)$ is analytic in the domain $(E_n - E_{n-m})_{(\varepsilon_i)}$, where the variables x_i ($i = n - m + 1, \dots, n$) preserve their sign ($0 < |x_i| < \infty$), and in each such domain

$$\int_{(E_n - E_{n-m})_{(\varepsilon_i)}} |x''|^{\alpha+jp} \sum_{\Sigma \beta = l+j} \left| \frac{\partial^{l+j} U(X)}{\partial x_1^{\beta_1} \dots \partial x_n^{\beta_n}} \right|^p dX \leq c \sum_{i=1}^{l-1} \sum_{i_{n-m+1} + \dots + i_n = i} \|\varphi_{i_{n-m+1}, \dots, i_n}^i\|_{W_p^{(l-i-1+r)}(E_{n-m})}^p, \quad j = 1, 2, \dots$$

The definitions and theorems stated above can be transferred to the case of domains with sufficiently smooth boundary, as was done in our preceding paper ⁽⁴⁾ for $l = 1$, $n = 2$. We shall restrict ourselves to formulating the following theorem, which is a direct development of the main theorem of paper ⁽⁴⁾.

Theorem 3*. Let the function $U(X)$ be defined in a domain G with boundary $\Gamma \in C^{(2)}$, and belong to $W_p^{(1)}(G_\delta)$ for every $\delta > 0$ (where G_δ is the subdomain of G whose points are at a distance from Γ greater than δ), and let the weighted integral

$$D_{p,\sigma}^1(U, G) = \int_G \sigma(X) |\text{grad } U|^p dX$$

be finite (where $\sigma(X)$ is a bounded function comparable with the distance to the boundary, i.e. $c_1\rho^\alpha(X, \Gamma) \leq \sigma(X) \leq c_2\rho^\alpha(X, \Gamma)$; $c_1, c_2 > 0$; $-1 < \alpha < p - 1$). Then there exist boundary values of the function $U(X)$ on the boundary almost at every point $x \in \Gamma$ (under approach along the normal), and the boundary function $U(X)|_\Gamma = \varphi(x)$, being also the limit of $U(X)$ in the p -mean, satisfies the conditions:

- 1) $\varphi(x) \in L_p(\Gamma)$;
- 2)

$$A_{p,\alpha}(\varphi, \Gamma) = \int_\Gamma dx \int_\Gamma \frac{|\varphi(x) - \varphi(y)|^p}{|x - y|^{n-2+p-\alpha}} dy \leq cD_{p,\alpha}^1(U, G).$$

Conversely, a function $\varphi(x) \in W_p^{\frac{p-1-\alpha}{p}}(\Gamma)$ (i.e. $\varphi(x) \in L_p(\Gamma)$ and $A_{p,\alpha}(\varphi, \Gamma) < \infty$) can be extended to G in the class $W_{p,\sigma}^1(G)$, i.e. for the resulting function $U(X)$ the following assertions will be valid:

- a) $U(X)|_\Gamma = \varphi(x)$;
- b) $U(X) \in W_p^1(G_\delta)$;
- c) $D_{p,\sigma}^1(U, G) \leq cA_{p,\alpha}(\varphi, \Gamma)$.

In conclusion we shall present theorems confirming the mutual “adaptability” of the weighted classes and the spaces $W_p^{(r)}$ also from the point of view of the relations between them in E_n .

* For $p = 2$, Theorem 3 in a somewhat different form was proved in paper (5).

Theorem 4. The following imbedding holds

$$W_{p,\alpha}^l(E_n^0) \subset W_p^{l-\frac{\alpha}{p}}(E_n^0) \quad \text{for } 0 \leq \alpha \leq pl;$$

$$W_{p,\alpha}^l(E_n^0) \subset W_p^l(E_n^0) \quad \text{for } -1 < \alpha \leq 0.$$

Thus, membership in a weighted class guarantees membership in some class $W_p^{(r)}$. In this case the difference integrals entering into the definition of $W_p^{(r)}$ (or the norms of higher derivatives when r is an integer) are estimated in terms of the weighted integral $D_{p,\alpha}^l$.

Theorem 5. Let the function $U(X)$ belong to $W_{p,\alpha}^l(E_n - E_{n-1})$, and let the limiting values coincide:

$$\lim_{x_n \rightarrow +0} \frac{\partial^i U(X)}{\partial x_1^{i_1} \dots \partial x_n^{i_n}} = \lim_{x_n \rightarrow -0} \frac{\partial^i U(X)}{\partial x_1^{i_1} \dots \partial x_n^{i_n}}, \quad 0 \leq i_1 + \dots + i_n = i \leq l - 1.$$

Then, for $0 \leq \alpha < p - 1$,

$$U(X) \in W_p^{l-\frac{\alpha}{p}}(E_n).$$

We shall finish the exposition with a theorem on extension, relating to the theory of the spaces $W_p^{(r)}$. In proving it, a method was used that is characteristic of the entire work as a whole. This method consists in using, for imbedding theorems, the property of absolute continuity of functions possessing generalized derivatives on almost all straight lines of a prescribed direction inside the domain under consideration ⁽¹³⁾, and it relies, among other integral inequalities, especially on Hardy's inequality ⁽¹⁴⁾. In the extension theorems various representations of the extended function are used, based on the Poisson integral.

Theorem 6. Under the conditions of Theorem 2 there exists a function $V(X)$, defined in E_n , possessing the properties:

$$\begin{aligned} \text{a)} \quad & V(X) \in W_p^{(l-1+r+\frac{m}{p})}(E_n); \\ \text{b)} \quad & \|V(X)\|_{W_p^{(l-1+r+\frac{m}{p})}(E_n)} \leq \\ & \leq c \sum_{i=0}^{l-1} \sum_{i_{n-m+1}+\dots+i_n=i} \|\varphi_{i_{n-m+1}, \dots, i_n}^i\|_{W_p^{l-i-1+r}(E_{n-m})}^p; \\ \text{c)} \quad & \left. \frac{\partial^i V(X)}{\partial x_{n-m+1}^{i_{n-m+1}} \dots \partial x_n^{i_n}} \right|_{x''=0} = \varphi_{i_{n-m+1}, \dots, i_n}^i(x'). \end{aligned}$$

Remark. The present work arose in the seminar of S. M. Nikol'skii, V. I. Kondrashov, and L. D. Kudryavtsev at the Mathematical Institute of the Academy of Sciences of the USSR. Theorems 1, 4, and 6 were obtained independently and simultaneously, by another method and in a somewhat different form, by S. V. Uspenskii ⁽¹⁵⁾.

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