I. M. Melnik

1960

SovietRxiv

Abstract

Full Text

I. M. Melnik

Indices of Branch Points of a Multivalued Function

(Presented by Academician P. S. Aleksandrov, May 11, 1960)

§ 1. In M. Morse' s monograph (^{1}), from the values prescribed on the contour of an analytic or pseudoanalytic function, relations are derived between the number of critical points of the corresponding function in the domain, the connectivity of the domain, and the boundary index of the real part of the function prescribed on the contour. In the work of F. D. Gakhov and Yu. M. Krikunov (^{2}), the results of M. Morse's monograph are generalized and refined in various directions. Some generalizations of the results of F. D. Gakhov and Yu. M. Krikunov were obtained in the works of T. A. Kolomiitseva (^{3}) and A. I. Povolotskii (^{4}). In the works mentioned, either single-valued analytic functions (corresponding to pseudoanalytic ones) were considered, or analytic functions having poles and such logarithmic branch points that the real or imaginary part of the function under investigation is single-valued in the domain studied.

In the present note, topological relations are derived for an analytic function $f(z)$ (corresponding to a pseudoanalytic one) having, in the given domain, a finite number of algebraic and transcendental branch points and a finite number of poles; moreover, the real and imaginary parts of $f(z)$ are multivalued functions in the domain under consideration.

Let G be a domain bounded by α Jordan curves $(\Gamma_1, \Gamma_2, \dots, \Gamma_\alpha) = \Gamma$. Since an infinite domain can be mapped one-to-one and with preservation of orientation onto a finite domain, in our investigation we shall restrict ourselves to the case of a finite domain. The results obtained for a finite domain automatically extend to the case of an infinite domain.

Definition 1. A boundary curve Γ_j will be called **external** if each of its points can be joined to the point at infinity by a Jordan curve lying entirely outside G.

Definition 2. An interior point a of the domain G will be called an **ordinary point** of the function $f(z)$ if $f(z)$ is analytic at the point a and $f'(a) \neq 0$. If $f'(z)$ vanishes or does not exist at the point a, then the point a will be called a **critical point** of the function $f(z)$.

§ 2. Let the function $f(z)$ in a neighborhood of an interior point a have the form

$$f(z) = \sum_{j=1}^{m} g_j(z)(z-a)^{q_j} + (z-a)^n \psi(z) + C, \tag{1}$$

where q_j are real numbers distinct from integers; n is an integer different from zero; the functions $g_j(z)$ and $\psi(z)$ are analytic at the point a, with $g_j(a) \neq 0$, $\psi(a) \neq 0$; C is a complex constant. The functions $g_j(z)$ satisfy the condition

$$\operatorname{Im} g_j(x + i\beta_0) = 0, \quad x \leq \alpha_0 \quad (j = 1, 2, \ldots, m); \tag{2}$$

$$a = \alpha_0 + i\beta_0, \quad -\pi \leq \arg(z-a) \leq \pi.$$

We shall assume the domain G to be cut along the line L ($t = x + i\beta_0$, $x \leqslant \alpha_0$), joining the point a with the outer boundary curve. Taking into account conditions (2), it is easy to show that

$$u^+(x, \beta_0) - u^-(x, \beta_0) = 0 \quad \text{on } L \quad (u(x,y) = \operatorname{Re} f(z)).$$

It follows from this that the function $u(x,y)$ is harmonic in the neighborhood of the point a cut along L and is continuously continuable through L everywhere except, possibly, at the point a. Putting $s = \min q_j$ $(j = 1, 2, \ldots, m)$, we have

$$f(z) = (z-a)^s g(z) + (z-a)^n \psi(z) + C, \qquad g(a) \neq 0.$$

Denote $p = E(s)$, where $E(s)$ means the greatest integer not exceeding s. If $s < n$ and $s - p = 1/2$, then, in the consideration, two cases are possible:

A. Among the numbers q_j there are some such that $\cos \pi q_j \neq 0$, and the least of these numbers q_k satisfies the condition $q_k < n$.

B. $\cos \pi q_j = 0$ for all $q_j < n$. In this case it is assumed that $\operatorname{Re} \psi(a) \neq 0$.

Lemma 1. *Let $z = a$ be a branch point of the function $f(z)$, in a neighborhood of which $f(z)$ has a representation of the form (1). There exists a sufficiently small number r_0 such that the increment $P(\gamma_0)$ of the boundary index of the function $u(x,y) = \operatorname{Re} f(z)$ over the circle $\gamma_0(|z-a| = r_0)$, with respect to the domain $|z-a| > r_0$, is computed by the formulas*

for $s < n$

$$P(\gamma_0) = p + E\{2(s-p)\}, \quad \text{if } s - p \neq 1/2;$$

if $s - p = 1/2$, then

$$P(\gamma_0) = p + \frac{1}{2}\left[1 \mp (-1)^p \operatorname{sign}(sq_k \cos \pi q_k \, g_k(a))\right] \quad \text{in case A;}$$

$$P(\gamma_0) = p + \frac{1}{2}\left[1 \mp (-1)^{n+p} \operatorname{sign}(ns \operatorname{Re} \psi(a))\right] \quad \text{in case B;}$$

$$P(\gamma_0) = n \quad \text{for } s > n.$$

The upper sign is to be taken when $g(a) > 0$*, the lower one when* $g(a) < 0$*.*

Obviously, if in relation (1) all the numbers q_j are rational, then the point a is an algebraic branch point of the function $f(z)$. If among the q_j there are irrational ones, then the branch point a belongs to the category of transcendental ones. In the case where $g_j(z) \equiv 0$ for $j = 1, 2, \ldots, m$, the point a for $n > 0$ is a zero of multiplicity $n - 1$ of the function $f'(z)$; for $n < 0$ the point a will be a pole of order n of the function $f(z)$.

§ 3. Let, in a neighborhood of the point a, the function $f(z)$ have the representation

$$f(z) = g(z)(z-a)^s \ln^q(z-a) + (z-a)^n \psi(z) + C, \tag{3}$$

where s is a real number; n and q are integers; the functions $g(z)$ and $\psi(z)$ are analytic at the point a, with $g(a) \neq 0$ and $\psi(a) \neq 0$; if $q \neq 0$ or s is not an integer, then the function $g(z)$ satisfies conditions (2); $-\pi \leqslant \arg(z-a) \leqslant \pi$, $0 \leqslant \arg \ln(z-a) \leqslant 2\pi$.

By elementary operations one can show that the function $u(x, y) = \operatorname{Re} f(z)$ is continuously continuable through the cut L everywhere except, possibly, at the point a.

Lemma 2. *The increment* $P(\gamma_0)$ *of the boundary index of the function* $u(x, y)$ *over the circle* γ_0 *of sufficiently small radius* r_0*, with respect to the domain* $|z - a| > r_0$*, is computed as follows:*

for $s \leqslant n$

$$P(\gamma_0) = p + E\{2(s-p)\}, \quad \text{if } s - p \neq 1/2;$$

$$P(\gamma_0) = p + \frac{1}{2}[1 - \operatorname{sign} q], \quad \text{if } s - p = 1/2 \text{ and } q \neq 0;$$

if $s > n$, then

$$P(\gamma_0) = n.$$

§ 4. Suppose that in a neighborhood of the point a

$$f(z) = \psi(z)\ln^q(z-a) + g(z)(z-a)^s + C, \tag{4}$$

where s and q mean the same as in the preceding paragraph; the functions $\psi(z)$ and $g(z)$ are analytic at the point a; $\psi(z)$ satisfies conditions (2), with $\psi(a) \neq 0$; if s is not an integer, then $g(z)$ also satisfies conditions (2), and, for $g(z) \neq 0$, $g(a) \neq 0$; $-\pi \leq \arg(z-a) \leq \pi$, $0 \leq \arg\ln(z-a) \leq 2\pi$, $q \neq 0$.

Lemma 3. If the function $f(z)$ in a neighborhood of the point a has a representation of the form (4), then the contribution of the boundary index of the function $u(x,y) = \operatorname{Re} f(z)$ from the circle γ_0, relative to the domain $|z-a| > r_0$, is computed by the formulas:

if $s > 0$,

$$P(\gamma_0) = 0;$$

for $s < 0$

$$P(\gamma_0) = p + \mathrm{E}\{2(s-p)\}, \qquad \text{if } s - p \neq \tfrac{1}{2};$$

$$P(\gamma_0) = p + \frac{1}{2}\left[1 \mp (-1)^{p+q}\operatorname{sign} q\psi(a)\right], \qquad \text{if } s - p = \tfrac{1}{2},$$

where the upper sign should be taken when $g(a) > 0$, the lower sign when $g(a) < 0$.

Definition. Let the function $u(x,y)$ be harmonic (or pseudoharmonic) in a neighborhood of the point a cut along L, be continuously extendable through L everywhere except, possibly, at the point a; let $P(\gamma_0)$ be the contribution of its boundary index from the circle γ_0 ($|z-a| = r_0$) relative to the domain $|z-a| > r_0$. The number $I(a) = 1 - P(\gamma_0)$ will be called the **index** of the point a of the function $u(x,y)$.

§ 5. Suppose that in G there are m critical points a_j of the function $f(z)$, in a neighborhood of each of which $f(z)$ has one of the representations considered in the preceding paragraphs; the function $u(x,y) = \operatorname{Re}(z)$ is harmonic in the domain G, cut along the lines L_j ($t_j = x + i\beta_j$, $x \leq \alpha_j$, $a = \alpha_j + i\beta_j$), is continuous in $\overline{G}$ everywhere except, possibly, at the points a_j, and satisfies on Γ any of the boundary conditions A, B, C ((1), p. 60).

Theorem. Let I be the boundary index of the function $u(x,y)$ along the contour Γ relative to G. Then the relation

$$\sum_{j=1}^{m} I(a_j) = 2 - \alpha + I, \tag{5}$$

holds, where $I(a_j)$ denotes the index of the point a_j of the function $u(x, y)$.

Proof. Denote by G_0 the domain obtained from G by deleting closed circular neighborhoods D_j ($|z - a_j| \leq r_j$) of all critical points a_j, of radii r_j so small that the neighborhood D_j contains no critical points of the function u other than the point a_j; denote by Γ_0 the boundary of the domain G_0; denote the boundary of the neighborhood D_j by γ_j. Let $u_0(x, y)$ be the function defined by the values of $u(x, y)$ in $\overline{G}_0$. Write the fundamental relation ((1), p. 55, formula 13.1) for u_0 in G_0. Since in G_0 the function u_0 has no critical points, we have

$$2 - \alpha_0 + I_0 = 0, \tag{6}$$

where α_0 is the connectivity of the domain G_0; I_0 is the boundary index of the function u_0 on Γ_0 relative to G_0. Obviously,

$$\alpha_0 = m + \alpha, \qquad I_0 = \sum_{i=1}^{m} P(r_i) + I.$$

Substituting the expressions for α_0 and I_0 into (6), we obtain relation (5).

Remark. The results of the present paper extend to pseudoanalytic functions that are obtained from analytic functions by a homeomorphic and orientation-preserving change of the argument.

Rostov-on-Don
State University

Received
30 I 1960

REFERENCES

1. M. Morse, *Topological Methods in the Theory of Functions of a Complex Variable*, IL, 1951.
2. F. D. Gakhov, Yu. M. Krykunov, *Izv. AN SSSR, Ser. Mat.*, **20**, 207 (1956).
3. T. A. Kolomyitseva, *Izv. Vyssh. Uchebn. Zaved., Matematika*, No. 3 (10), 97 (1959).
4. A. I. Povolo kii, *DAN*, **129**, No. 2, 265 (1959).