HYDRAULICS

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1960

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Abstract

Full Text

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ON THE PROBLEM OF DETERMINING THE PRESSURE FUNCTION IN STRATA OF VARIABLE THICKNESS UNDER ELASTIC CONDITIONS

(Presented by Academician P. Ya. Kochina on 1 VI 1960)

The unsteady motion of a homogeneous fluid according to the linear law of filtration is considered in a circular stratum of unit radius ($R = 1$), which is exploited by means of n arbitrarily located wells.

It is required to determine the pressure $P(r, \theta, t)$ at any moment of time (after the well is put into operation) at any point of the stratum, if on the supply contour $r = 1$ the initial pressure $P_0 = \text{const}$ is maintained and the thickness of the stratum $H(r, \theta) > 0$ satisfies the equation

$$\Delta\sqrt{H} - \alpha\sqrt{H} = 0, \tag{1}$$

where α is any real number and Δ is the Laplace operator. In addition, the function $P(r, \theta, t)$ at the well points with coordinates (δ_i, θ_i), which are regarded as vertical linear sinks or sources, must have a singularity of the type of a heat source or sink of constant strength Q_i.

If one introduces a new function u, related to P by the relation

$$P = P_0 + \frac{u}{\sqrt{H}}, \tag{2}$$

and takes (1) into account, then the problem is mathematically formulated as follows ([1,2]): it is required to solve the differential equation

$$\Delta u - \alpha u = \frac{1}{\omega}\frac{\partial v}{\partial t} \tag{3}$$

subject to the following initial and boundary conditions:

$$u(r, \theta, 0) = 0; \tag{4}$$

$$u(1, \theta, t) = 0; \tag{5}$$

$$\lim_{\rho \to 0} \rho \frac{\partial}{\partial \rho} \left(\frac{u}{\sqrt{H}} \right) = \text{const}, \tag{6}$$

where ω is the piezoconductivity coefficient, $\rho = \sqrt{r^2 - 2r\delta_i \cos(\theta - \theta_i) + \delta_i^2}$ is the distance from the center of the well.

First the case is considered in which the stratum is exploited by only one fixed well with coordinates $(\delta, 0)$. To solve the formulated problem, the Laplace transform with respect to the variable t is used ([4])

$$L[u(r, \theta, t)] = \bar{u}(r, \theta, q) = \int_0^\infty e^{-qt} u(r, \theta, t) \, dt.$$

The solution of equation (3), after applying the Laplace transform to it with allowance for the initial condition (4), has the form

$$\bar{u}(r, \theta, q) = AK_0\left(\sqrt{s(r^2 - 2r\delta \cos \theta + \delta^2)}\right) + \sum_{m=0}^{\infty} B_m I_m(r\sqrt{s}) K_m(\sqrt{s}) \cos m\theta, \tag{7}$$

where $s = \alpha + q/\omega$; I_m and K_m are the modified Bessel functions of the first and second kind of order m.

The arbitrary constants A, B_m are determined according to condition (5) and the condition of constancy of the rate Q, which is given by the formula

$$\bar{Q} = \frac{Q}{q} = ? - \frac{k}{\mu} \lim_{r_c \to 0} \int_0^{2\pi} \left[H\rho \frac{\partial}{\partial \rho} \left(\frac{\bar{u}}{\sqrt{H}} \right) \right]_{\rho = r_c} d\varphi,$$

where r_c is the well radius, k is the permeability coefficient, and μ is the dynamic viscosity of the liquid.

Then the transform of the function $u(r, \theta, t)$ satisfying the boundary conditions (5), (6) will be

$$\bar{u}(r, \theta, q) = \frac{\mu Q}{2\pi k \sqrt{H_c}} \frac{1}{q} \left[K_0\left(\sqrt{s(r^2 - 2r\delta \cos \theta + \delta^2)}\right) - \right.$$

$$\left. - \sum_{m=0}^{\infty} \varepsilon_m \frac{I_m(\delta\sqrt{s}) K_m(\sqrt{s})}{I_m(\sqrt{s})} I_m(r\sqrt{s}) \cos m\theta \right], \tag{8}$$

where $\varepsilon_m = 1$ for $m = 0$, $\varepsilon_m = 2$ for $m \geqslant 1$.

By the inversion theorem we have

$$L^{-1}[\bar{u}(r,\theta,q)] = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{qt}\bar{u}(r,\theta,q)\,dq. \tag{9}$$

It is easy to show that the function $\bar{u}(r,\theta,q)$ has no singularities other than the countable set of poles: $q = 0$, $q = -\omega(\beta_1^2+\alpha), -\omega(\beta_2^2+\alpha), ...$, where $\beta_1, \beta_2, ...$ are the roots of the equation $J_m(\beta) = 0$ $(s = \alpha + q/\omega = -\beta^2)$. Therefore the line $(\gamma - i\infty, \gamma + i\infty)$ in (9) may be replaced by a circle Γ containing inside it all the poles of the integrand.

Finding the residues with respect to these poles and summing them, for all $r < \delta$, we obtain

$$u(r,\theta,t) = \frac{\mu Q}{2\pi k\sqrt{H_c}}\Big[K_0\Big(\sqrt{\alpha(r^2 - 2r\delta\cos\theta + \delta^2)}\Big) -$$

$$-\sum_{m=0}^{\infty}\varepsilon_m\frac{I_m(\delta\sqrt{\alpha})K_m(\sqrt{\alpha})}{I_m(\sqrt{\alpha})}I_m(r\sqrt{\alpha})\cos m\theta -$$

$$-2\sum_{m=0}^{\infty}\varepsilon_m\cos m\theta\sum_{j=1}^{\infty}\frac{J_m(\delta\beta_j)J_m(r\beta_j)}{(\beta_j^2+\alpha)J_{m'}^2(\beta_j)}e^{-\omega(\beta_j^2+\alpha)t}\Bigg]; \tag{10}$$

by virtue of the symmetry of expression (10) with respect to r and δ, it is also valid for $r > \delta$. Here the following identities were used ([3]):

$$I_m(iz) = i_m^{mJ}(z), \qquad I_m'(iz) = i^{m-1}J_m'(z),$$

$$K_m(iz) = \frac{1}{2}\pi(-i)^{m+1}\left[J_m(z) - iY_m(z)\right],$$

$$J_m(\alpha z)Y_m'(\alpha z) - Y_m(\alpha z)J_m'(\alpha z) = \frac{2}{\pi\alpha z},$$

where J_m, Y_m are Bessel functions of order m of the first and second kind.

By virtue of the linearity of equation (3), as well as of conditions (4)–(6), the superposition principle can be applied for n wells with coordinates (δ_i, θ_i). Then, passing at the same time to $P(r,\theta,t)$ with allowance for (2), we obtain the final formula for determining the pressure function in a stratum that is operated by means of n arbitrarily placed wells:

$$P = P_0 + \frac{\mu}{2\pi k\sqrt{H}} \sum_{i=1}^{n} \frac{Q_i}{\sqrt{H_i}} \left[K_0\left(\sqrt{\alpha\,(r^2 - 2r\delta_i \cos(\theta-\theta_i) + \delta_i^2)} \right) - \right.$$

$$- \sum_{m=0}^{\infty} \varepsilon_m \frac{I_m(\delta_i\sqrt{\alpha})K_m(\sqrt{\alpha})}{I_m(\sqrt{\alpha})} I_m(r\sqrt{\alpha}) \cos m(\theta-\theta_i) -$$

$$\left. -2\sum_{m=0}^{\infty} \varepsilon_m \cos m(\theta-\theta_i) \sum_{j=1}^{\infty} \frac{J_m(\delta_i\beta_j)J_m(r\beta_j)}{(\beta_j^2+\alpha)J_m'^2(\beta_j)} e^{-\omega(\beta_j^2+\alpha)t} \right]. \tag{11}$$

The formula for determining the pressure function in a circular stratum with one central well, derived by V. Yu. Kim ([5]) for the case when $\sqrt{H(r)}$ is a harmonic function of one variable r, is obtained from (11) under quite special assumptions. Indeed, for this it is sufficient to put $n = 1$ in (11) and pass to the limit as $\delta_i \to 0,\ \alpha \to 0$. We obtain

$$P = P_0 - \frac{\mu Q}{2\pi k\sqrt{H \cdot H_0}} \left[\ln r + 2\sum_{j=1}^{\infty} \frac{J_0(r\beta_j)e^{-\omega\beta_j^2 t}}{\beta_j^2 J_1^2(\beta_j)} \right]. \tag{12}$$

We further note in passing that, according to the well-known thermohydrodynamic analogy, it may be considered that the solution found for the problem under study in filtration theory is simultaneously a solution of a related problem of heat-conduction theory ([6]).

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Received
25 V 1960

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Note: Figure translations are in progress. See original paper for figures.