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Abstract

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V. Yu. Krylov

ON SOME PROPERTIES OF THE DISTRIBUTION CORRESPONDING TO THE EQUATION

$$\frac{\partial u}{\partial t} = (-1)^{q+1}\frac{\partial^{2q} u}{\partial x^{2q}}$$

(Presented by Academician A. N. Kolmogorov on 27 II 1960)

As early as the 1920s, N. Wiener ([1,2]) considered a probability distribution in the space of continuous functions—the trajectories of Brownian particles—associated with the heat equation. This distribution has since been studied by many authors (see the bibliography in ([3])) and is usually called Wiener measure. I. M. Gelfand posed the problem of studying distributions analogous to Wiener measure that are associated with differential equations of a more general form ([3], *p*.87).

The present work is devoted to the study of certain properties of a distribution in the space $C[0,T]$ of functions $x(t)$, continuous on $[0,T]$, corresponding to the equation

$$\frac{\partial u}{\partial t} = (-1)^{q+1}\frac{\partial^{2q} u}{\partial x^{2q}}. \tag{1}$$

This distribution is defined, as usual, on cylindrical subsets (quasi-intervals) of the space $C[0,T]$. It is proved here that the mean, with respect to the distribution, of the functional

$$\exp\left\{-\int_0^T V[x(t)]\,dt\right\}$$

exists and is a solution of the equation

$$\frac{\partial u}{\partial t} = (-1)^{q+1}\frac{\partial^{2q} u}{\partial x^{2q}} - V(x)u \tag{2}$$

for a certain class of functions $V(x)$. Thus the existence of the distribution, or generalized measure P_{2q}, is proved already in the whole space $C[0,T]$ of continuous functions. It turns out that the total variation on $C[0,T]$ of the measure P_{2q} for $q>1$ is infinite. In the present work it is established that the measure P_{2q} is concentrated on a compact set in the sense that its total variation outside this compact set can be made arbitrarily small. In addition, for any $q>1$ the "arcsine law"known for Wiener measure is generalized ([3], $p.90$).

A measure in the space $C[0,T]$ of functions $x(t)$, continuous on the interval $[0,T]$ and satisfying the condition $x(0)=0$, is defined as follows. Divide the interval $[0,T]$ into n equal parts, each of length $\Delta = T/n$, by the points $t_k = kT/n$ $(k=0,\dots,n)$.

Let the set I_n of functions $x(t)\in C[0,T]$ satisfying the inequalities $a_k \le x(t_k) \le b_k$ $(k=1,\dots,n)$ be called a quasi-interval. Define the measure $P_{2q}\{I_n\}$ of the quasi-interval I_n by the formula

$$P_{2q}\{I_n\} = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{k=0}^{n-1} G(t_{k+1}-t_k,\ x_{k+1}-x_k)\, dx_{k+1}, \tag{3}$$

where $x_0=0;\ t_0=0;\ G(t,x)$ is the Green' s function

$$G(t,x) = t^{-1/2q} g(xt^{-1/2q}), \qquad g(y) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-\xi^{2q}+iy\xi}\, d\xi$$

of equation (1)*. For each integer $m\le n$, define the function

$$u_m(t,x) = \int_{-\infty}^{+\infty} \cdots \int \psi(x_0) \exp\left\{-\Delta \sum_{k=0}^{m-1} V(x_{k+1})\right\} \prod_{k=0}^{m-1} G(\Delta, x_{k+1}-x_k)\, dx_k,$$

where $x_m = x(t_m) = x(t) = x,\ t_m = m\Delta$.

Theorem 1. *Let $V(x)$ be a bounded smooth function. Then the sequence $u_m(t,x)$ converges uniformly in x on every finite interval to a limit $u(t,x)$, which is the solution of the Cauchy problem for equation (2) with initial condition $u(0,x)=\psi(x)$.*

The assertion of Theorem 1 follows directly from the compactness of the family of functions $u_m^\Delta(x) = u_m(t,x)$. The compactness of the family of functions $u_m^\Delta(x)$ is easily derived from the relation

$$u_{m+1}^\Delta(x) = e^{-\Delta V(x)} \int_{-\infty}^{+\infty} G(\Delta, x-\xi) u_m^\Delta(\xi)\, d\xi.$$

Theorem 2. *The measure $P_{2q}\{C\}$ has unbounded total variation for $q>1$.*

Proof. The measure $P_{2q}\{C\}$ is given by a density; therefore its total variation on $C[0,T]$ is equal to the integral over $C[0,T]$ of the modulus of the density

$$\mathop{\mathbf{V}}_{C} P_{2q} = \lim_{n\to\infty} \int_{-\infty}^{+\infty} \cdots \int \prod_{k=0}^{n-1} |G(\Delta, x_{k+1} - x_k)| \, dx_{k+1}.$$

Taking into account that the function $g(y)$ for $q > 1$ is sign-changing and that

$$\int_{-\infty}^{+\infty} g(y)\, dy = 1,$$

and consequently

$$\int_{-\infty}^{+\infty} |g(y)|\, dy > 1,$$

we obtain

$$\mathop{\mathbf{V}}_{C} P_{2q} = \lim_{n\to\infty} \left(\int_{-\infty}^{+\infty} |g(y)|\, dy \right)^n = \infty,$$

which proves Theorem 2.

Let t vary on the interval $[0,1]$. Consider the set $A_H^p \subset C[0,1]$ of continuous functions $x(t)$, equal to zero at $t = 0$, for each of which there exist points t_1 and t_2 on the interval $[0,1]$ such that

$$|x(t_1) - x(t_2)| > H\gamma |t_1 - t_2|^{1/p},$$

where H and γ are some positive numbers. The set complementary to A_H^p is, obviously, compact for any $p > 0$.

Theorem 3. *The total variation of the measure $P_{2q}\{A_H^p\}$ on the set A_H^p tends to zero as $H \to \infty$, if $1/p = 1/2q - \varepsilon$, where ε is any positive number.*

The proof of this theorem will rely on the following lemma, which we merely state.

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* A formula analogous to (3) for the case of the Schrödinger equation is found in the work of R. P. Feynman (4), and for equations of a more general form was given by L. V. Kobelev (unpublished).

Lemma. If the function $x(t) \in A_H^p$, then there exist integers m, i (with $m \leqslant 2^i$) such that

$$\left|x[m2^{-i}] - x[(m+1)2^{-i}]\right| > H2^{-i/p}, \tag{4}$$

for a suitable choice of $\gamma = \gamma(p)$.

We now pass to the proof of Theorem 3. By Lemma 1, for any function from the set A_H^p there exist integers m and i such that inequality (4) is satisfied. Therefore it is enough for us to estimate the variation of the measure P_{2q} on functions satisfying (4) for all $m \leqslant 2^i$ and all i. Obviously, this variation is equal to

$$\mathrm{V}_{A_H^p} P_{2q} = \sum_{i=0}^{\infty} 2^{i+1} \int_{H2^{-i/p}}^{\infty} \left|G(2^{-i},\xi)\right| d\xi = \sum_{i=0}^{\infty} 2^{i+1} \int_{H2^{\,i(1/2q-1/p)}}^{\infty} |g(y)|\, dy, \tag{5}$$

where $y = \xi 2^{i/2q}$. Since $|g(y)| < \exp\{-Cy\}$ for large y and for any $q \geqslant 1$ (see, for example, (5), p. 120), it follows that for $1/2q - 1/p = \varepsilon > 0$ the sum of the series (5) tends to zero as $H \to \infty$, which is what was asserted in Theorem 3.

Finally, let us consider one more property of the measure P_{2q}. For a Brownian particle the "arcsine law" is known. According to it, the probability $F(t_1)$ that a particle which is at the origin at the moment $t = 0$ is on the positive half-axis during the time T for a time not less than t_1 is equal to

$$F(t_1) = \frac{2}{\pi} \arcsin \sqrt{\frac{t_1}{T}}. \tag{6}$$

An unexpected circumstance is that this formula is valid for any $q \geqslant 1$.

Theorem 4. For any $q \geqslant 1$, the full measure $F(t_1)$ of the set of those trajectories $x(t) \in C[0,T]$ which, on the interval $[0,T]$, are positive during a time not less than t_1, is given by formula (6).

Proof. Obviously,

$$\Phi(T) = \int_{C[0,T]} \exp\left\{ - \int_{\substack{x(t)>0 \\ t \leqslant T}} dt \right\} dP_{2q}[x(t)] = \int_0^T e^{-t_1}\, dF(t_1). \tag{7}$$

Thus we need to find the value of the integral appearing on the left-hand side of equality (7). Let $\tilde{u}(t,x)$ be the solution of equation (2) with potential $V(x)$, equal to 1 for $x > 0$ and 0 for $x < 0$, satisfying the conditions

$$\lim_{x \to \pm\infty} \tilde{u}(t,x) = 0, \qquad \tilde{u}(0,x) = \delta(x).$$

The Laplace transform $\varphi(x)$ of the function $\tilde{u}(t,x)$,

$$\varphi(x) = \int_0^\infty \tilde{u}(t,x) e^{-Et}\, dt,$$

will satisfy the equation

$$(-1)^{q+1} \varphi^{(2q)}(x) = (V(x) + E)\varphi(x)$$

and the conditions

$$\lim_{x\to\pm\infty} \varphi(x) = 0, \qquad \varphi(+0) = \varphi(-0), \dots, \quad \varphi^{(2q-2)}(+0) = \varphi^{(2q-2)}(-0),$$

$$\varphi^{(2q-1)}(+0) - \varphi^{(2q-1)}(-0) = (-1)^q.$$

By virtue of the assertion of Theorem 1, we have

$$\int_0^\infty \Phi(T) e^{-ET}\, dT = \int_{-\infty}^{+\infty} \varphi(x)\, dx. \tag{8}$$

Introduce the notation $a_k = \sqrt[2q]{E}\,\varepsilon_k$ for $k = 1, \dots, q$ and $a_k = \sqrt[2q]{E+1}\,\varepsilon_k$ for $k = q+1, \dots, 2q$, where ε_k are the roots of the equation $\varepsilon^{2q} = (-1)^{q+1}$, numbered so that $\operatorname{Re}\varepsilon_k > 0$ for $k = 1, 2, \dots, q$.

It is easy to verify that, in this notation, the desired function $\varphi(x) = W^{-1}\sum_{k=q+1}^{2q} W_k \exp(a_k x)$ for $x > 0$, and $\varphi(x) = -W^{-1}\sum_{k=1}^{q} W_k \exp(a_k x)$ for $x < 0$, where W is the Vandermonde determinant of order $2q$ formed from a_k ($k = 1, 2, \dots, 2q$), and W_k is obtained from W by replacing the k-th column by the vector $\{0, \dots, 0, (-1)^q\}$. The integral appearing on the right-hand side of equality (8) is equal to

$$\int_{-\infty}^{+\infty} \varphi(x)\, dx = -\sum_{k=1}^{q} \frac{W_k}{a_k W} =$$

$$= \frac{(-1)^{q+1}}{W} \sum_{k=1}^{2q} \frac{(-1)^k}{a_k} \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ a_1 & \dots & a_{k-1} & a_{k+1} & \dots & a_{2q} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_1^{2q-2} & \dots & a_{k-1}^{2q-2} & a_{k+1}^{2q-2} & \dots & a_{2q}^{2q-2} \end{vmatrix} = \frac{(-1)^{q+1}}{W \prod_{i=1}^{2q} a_i} \begin{vmatrix} 1 & \dots & 1 \\ a_1 & \dots & a_{2q} \\ \cdot & \cdot & \cdot \\ a_1^{2q-2} & \dots & a_{2q}^{2q-2} \\ \prod_{i\neq 1}^{2q} a_i & \dots & \prod_{i\neq 2q}^{2q} a_i \end{vmatrix} =$$

$$= (-1)^q \left(\prod_{i=1}^{2q} a_i\right)^{-1} = \frac{1}{\sqrt{E(1+E)}}$$

by the definition of the quantities a_k. Inverting the Laplace transform (8), we obtain

$$\Phi(T) = \frac{1}{\pi} \int_0^T \frac{e^{-t_1}\, dt_1}{\sqrt{t_1(T - t_1)}} = \int_0^T e^{-t_1}\, dF(t_1),$$

which was required to be proved.

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Note: Figure translations are in progress. See original paper for figures.